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# Spanning Minimal Surfaces

W. Fischer and E. Koch

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# Spanning minimal surfaces

By W. Fischer and E. Koch
*Institut für Mineralogie, Petrologie und Kristallographie,*
*(und Wissenschaftliches Zentrum für Materialwissenschaften),*
*Philipps-Universität Marburg, D-35032 Marburg, Germany*

Spanning minimal surfaces are 3-periodic minimal surfaces which contain straight lines, i.e. axes of 2-fold symmetry. We have used crystallographic knowledge of the space groups and of the corresponding arrangements of 2-fold axes involved for the derivation of new surfaces of this type and for their systematic description. Complete information on spanning minimal surfaces without self-intersections is given with respect to their symmetry and topology, together with some new results on spanning minimal surfaces with self-intersections along straight lines.

## 1. Introduction

A minimal surface in $R^3$ is usually defined as a surface, the mean curvature $H = \frac{1}{2}(k_1 + k_2)$ of which equals zero at each of its points. Accordingly, the two main curvatures $k_1$ and $k_2$ are equal in magnitude but opposite in sign, and the surrounding of each point is saddle shaped. As a consequence, there exist two tangential directions with zero curvature at each point. In the exceptional case of $k_1 = k_2 = 0$ the curvature is zero in any direction and the saddle has more than two valleys and two ridges. Such points, known as *flat points*, are important for the numerical calculation of minimal surfaces, something which is often difficult. Except for the plane, flat points are isolated points.

Well-known examples of minimal surfaces are the catenoid and the helicoid. The catenoid is aperiodic, and the helicoid periodic in one direction. The first examples of 3-periodic minimal surfaces were given by Schwarz (1890), who originally presented his results in 1867 to the Berlin Academy, and by his pupil Neovius (1883). Later, remarkable pioneering work in the field was done by Stessmann (1934) and especially by Schoen (1970) (cf. Karcher 1989). Schoen made use of the two reflection principles of Schwarz (1890). (1) If part of the boundary of a minimal-surface patch is a straight line, then the minimal surface is smoothly continued across this line by 2-fold rotation. (2) If a minimal-surface patch meets a plane at right angles, then the minimal surface is smoothly continued across this plane by mirror reflection.

More recently, crystallographers became interested in 3-periodic minimal surfaces because of their relation to certain crystal structures (Andersson 1983; Andersson *et al.* 1984; Hyde & Andersson 1984, 1985; Mackay 1985). Based on Schwarz's first reflection principle and on crystallographic knowledge of the mutual orientation and position of 2-fold rotation axes in space groups (International Tables for Crystallography (Hahn) 1983), a systematic treatment of 3-periodic minimal surfaces containing straight lines was developed and new surfaces were derived (Fischer & Koch 1987, 1989*a*–*c*, 1990, 1992, 1995, 1996; Koch & Fischer 1988, 1989*a*, *b*, 1990, 1993*a*, *b*; Koch 1995).

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A 3-periodic minimal surface in $R^3$ is either free of self-intersections or may intersect itself in a more or less complicated way. Each 3-periodic minimal surface without self-intersections is orientable, i.e. two-sided, and subdivides $R^3$ into two infinite connected disjunct regions. These two regions are not simply connected and they interpenetrate each other in a complicated way. They are known as the labyrinths of the surface. If its two labyrinths are different, the symmetry of a surface is described by a space group $G$ (219 or 230 types according to whether the left and right enantiomers are considered to be the same or different). If they are congruent, i.e. if there exist symmetry operations mapping one labyrinth onto the other, then the surface is termed a (3-periodic) *minimal balance surface*. Its symmetry is more adequately described by a group–subgroup pair $G$–$S$ of space groups with index 2 (1156 or 1191 types), where $G$ is the group of all symmetry operations which map the surface onto itself, while $S$ contains only those symmetry operations which map each labyrinth onto itself. The coset $G\backslash S$ then consists of all symmetry operations interchanging the two labyrinths (Fischer & Koch 1987). This symmetry situation may also be described with the aid of black–white space groups (Mackay & Klinowski 1986). According to the first reflection principle of Schwarz (1890), a minimal surface without self-intersections and which contains straight lines must be a minimal balance surface.

On the other hand, a 3-periodic minimal surface with self-intersections may be either orientable or non-orientable, i.e. one-sided like a Möbius strip or a Klein bottle. Only the special case of self-intersections along straight lines is considered in this paper. Regardless of its orientability, such a surface also subdivides $R^3$ into spatial subunits, but these may be either two or more 3-periodic labyrinths, or infinitely many 2-periodic labyrinths ('flat labyrinths'), 'tubes' or 'polyhedra'.

Schwarz (1894) was the first to pay special attention to 3-periodic minimal surfaces containing 2-fold axes. A characteristic feature of such surfaces is that in many cases their existence is guaranteed without sophisticated individual proof because of the general solution of the Plateau problem for the spanning of frames by minimal surfaces (cf. textbooks on minimal surfaces, e.g. Nitsche 1989; Dierkes *et al.* 1992). In what follows, only 3-periodic minimal surfaces containing straight lines will be considered. For the sake of brevity such surfaces will be called *spanning minimal surfaces*. Note that spanning minimal surfaces without self-intersections are special cases of minimal balance surfaces. The gyroid surface derived by Schoen (1970) is an example of a minimal balance surface which is not a spanning surface. Usually a non-periodic surface in $R^3$ is topologically characterized by three integers $(\varepsilon, r, \chi)$: the orientability character $\varepsilon$ (+1 for an orientable, −1 for a non-orientable surface), the number $r$ of boundary curves of the surface and its *Euler characteristic* $\chi$. The Euler characteristic $\chi = f - e + v$ can be derived from any arbitrary tiling of the surface, where $f$ is the number of faces (tiles), $e$ the number of edges and $v$ the number of vertices in the tiling.

3-periodic spanning minimal surfaces may be similarly characterized: $r = 0$ holds for any such surface, while $\varepsilon = +1$ must hold only for surfaces without self-intersection. As the Euler characteristic $\chi$ becomes infinite for any 3-periodic surface, it is necessary to refer $\chi$ not to the entire surface, but to calculate $f$, $e$ and $v$ only for one primitive unit cell.

## 2. Spanning minimal surfaces without self-intersections

### (a) Symmetry restrictions

As stated above, the symmetry of a minimal balance surface is adequately described by a group–subgroup pair $G$–$S$ of space groups with index 2. Any symmetry operation $g \in G$ maps the surface as a whole onto itself. If $g$ belongs to the coset $G\backslash S$ of $S$, then $g$ interchanges the two sides of the surface and the two labyrinths. All fixed points of $g \in G\backslash S$ must therefore lie on the surface. As an intersection-free minimal surface cannot contain embedded mirror planes or 3-, 4- or 6-fold rotation axes, all types of space-group pairs $G$–$S$ where $G\backslash S$ contains mirror reflections, 3-, 4- or 6-fold rotations or 6-fold roto-inversions are incompatible with minimal balance surfaces (Fischer & Koch 1987). This rules out 609 of the 1156 types of space-group pairs $G$–$S$ (members of enantiomorphic pairs are not counted separately). The possible symmetries for intersection-free spanning minimal surfaces are further restricted because the coset $G\backslash S$ must contain 2-fold rotations. There remain 352 types of space-group pairs.

A closer inspection of these space-group pairs results in 52 patterns of 2-fold axes, 52 line configurations which may be used as frameworks to be spanned by such minimal surfaces (Koch & Fischer 1988, 1993*b*). Eighteen of these line configurations consist of 2-fold axes in three linearly independent directions which intersect each other such that the entire line configuration forms a three-dimensionally connected net, with skew polygons as meshes. Three further line configurations are made up of three-dimensionally connected nets of 2-fold axes and additional 2-fold axes not intersecting the axes of the three-dimensional net. Twelve line configurations consist of parallel plane nets of 2-fold axes. In nine cases all nets are congruent, whereas the three remaining cases each contain two different kinds of alternating nets. Seven further line configurations also consist of plane nets, but either perpendicular to two directions (interpenetrating without intersection) or with additional isolated 2-fold axes or both. The remaining twelve line configurations contain only non-intersecting 2-fold axes.

A line configuration may be spanned by a minimal surface. Depending on the connectivity of the line configuration, different kinds of surface patches may be used as starting units for the generation of infinite minimal surfaces by application of Schwarz's first reflection principle.

### (b) Disc-like surface patches

The 18 three-dimensionally connected line configurations give rise to the simplest kind of surface patches, because they contain skew polygons, i.e. closed circuits of 2-fold axes, which may be spanned disc-like (cf. figure 1). According to the general solution of the Plateau problem, such a spanning by a patch of a minimal surface always exists (cf. Nitsche 1989; Dierkes *et al.* 1992). Therefore, any skew polygon found in a line configuration may be used to generate a 3-periodic minimal surface, but self-intersection will necessarily occur if too large a skew polygon is chosen. In order to avoid self-intersection two conditions must be fulfilled.

(i) Each vertex angle of the skew polygon must be chosen to be as small as possible. Especially, no angle may be larger than 90°.

(ii) None of the 2-fold axes out of the line configuration, i.e. no 2-fold axis not belonging to $S$, may run through the skew polygon.

Table 1 contains the complete list of all skew polygons which give rise to 3-periodic





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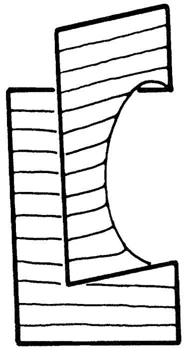

Figure 1. Spanned 8-gon, surface patch of an HS2 surface.

minimal surfaces without self-intersections (Fischer & Koch 1987; Koch & Fischer 1988, 1993*a*). Completeness has been assured by comparing the length of the polygon edges with the length of all 2-fold axes of the line configuration within one asymmetric unit of $G$. For each type of skew polygon the symmetry of the generated infinite surface is described in column 1 by a group-subgroup pair $G$–$S$. If $S$ refers to a larger conventional unit cell than $G$, the cell enlargement is specified in parentheses. $2a$, $2b$ and $2c$ mean a doubling of the corresponding lattice parameter, whereas $v$ stands for the transformation $\boldsymbol{a}' = \boldsymbol{a} - \boldsymbol{b}$, $\boldsymbol{b}' = \boldsymbol{a} + \boldsymbol{b}$. Further columns describe each generating surface patch: column 2 shows its site symmetry with respect to $G$ and to $S$; the number and positions of its vertices are given in columns 3 and 4. The generated infinite minimal surface is identified by a symbol in the last column. These symbols, which have been used in previous publications, come from different sources and therefore reflect different properties of the surfaces or their labyrinths. Clear and simple nomenclature has not yet been developed.

Table 1 shows that the same minimal surface may be generated with the aid of different skew polygons referring to different space-group pairs. In such a case, there always exists a smallest surface patch, and all further surface patches result from uniting some of the smallest patches. As a consequence, there exists a subgroup relationship between the corresponding space-group pairs. Disc-like surface patches may be used directly as faces of a tiling on the infinite minimal surface for the calculation of its Euler characteristic:

$$\chi = f - e + v. \tag{2.1}$$

$f$, $e$ and $v$ are then, respectively, the numbers of skew polygons, of edges, and of vertices in the tiling, referred to a primitive unit cell of $S$. If $e_p$ is the number of edges of one skew polygon and $v_i$ is the multiplicity of the $i$th kind of symmetrically equivalent vertex (also referred to a primitive unit cell of $S$), the formula for the Euler characteristic may be rewritten as (Fischer & Koch 1989*c*)

$$\chi = f(1 - \tfrac{1}{2}e_p) + \sum_i v_i. \tag{2.2}$$

### (*c*) *Catenoid-like surface patches*

If a line configuration consists of congruent parallel plane nets and if the polygon centers of neighbouring nets lie directly above each other, then a ring-like surface patch similar to a catenoid may span two such polygons (cf. figure 2). A well-known

Table 1. *Disc-like surface patches generating 3-periodic minimal surfaces without self-intersections*

| space-group pair $G$-$S$ | disc-like surface patch: site sym. | polygon | vertices | | | | minimal surface |
|---|---|---|---|---|---|---|---|
| $Im\bar{3}m$–$Pm\bar{3}m$ | $m.m2$–$m.m2$ | 4-gon | $8c\ .\bar{3}m\ \frac14\frac14\frac14$ | $12d\ \bar{4}m.2\ \frac12 0\frac14$ | $8c\ .\bar{3}m\ \frac34\frac14\frac14$ | $12d\ \bar{4}m.2\ \frac12\frac14 0$ | **P** |
| | $4m.m$–$4m.m$ | 8-gon | $8c\ .\bar{3}m\ \frac14\frac14\frac14$ | $12d\ \bar{4}m.2\ \frac12\frac14 0$ | $8c\ .\bar{3}m\ \frac34\frac14\frac14$ | $12d\ \bar{4}m.2\ \frac34\frac12 0$ | **C(P)** |
| | | | $8c\ .\bar{3}m\ \frac34\frac34\frac14$ | $12d\ \bar{4}m.2\ \frac12\frac34 0$ | $8c\ .\bar{3}m\ \frac14\frac34\frac14$ | $12d\ \bar{4}m.2\ \frac14\frac12 0$ | |
| $Pn\bar{3}m$–$Fd\bar{3}m(2a)$ | $..m$–$..m$ | 4-gon | $6d\ \bar{4}2.m\ 00\frac12$ | $12f\ 2.22\ \frac14 0\frac12$ | $4c\ .\bar{3}m\ \frac14\frac14\frac34$ | $12f\ 2.22\ 0\frac14\frac12$ | **D** |
| | $.3m$–$.3m$ | 12-gon | $12f\ 2.22\ \frac14 0\frac12$ | $6d\ \bar{4}2.m\ \frac12 0\frac12$ | $12f\ 2.22\ \frac12 0\frac14$ | $4c\ .\bar{3}m\ \frac34\frac14\frac14$ | **C(D)** |
| | | | $12f\ 2.22\ \frac12\frac14 0$ | $6d\ \bar{4}2.m\ \frac12\frac12 0$ | $12f\ 2.22\ \frac14\frac12 0$ | $4c\ .\bar{3}m\ \frac14\frac34\frac14$ | |
| | | | $12f\ 2.22\ 0\frac12\frac14$ | $6d\ \bar{4}2.m\ 0\frac12\frac12$ | $12f\ 2.22\ 0\frac14\frac12$ | $4c\ .\bar{3}m\ \frac14\frac14\frac34$ | |
| $Fd\bar{3}m$–$F\bar{4}3m$ | $2.mm$–$2.mm$ | 4-gon | $16c\ .\bar{3}m\ \frac18\frac18\frac18$ | $16d\ .\bar{3}m\ \frac38\frac18\ {-\frac18}$ | $16c\ .\bar{3}m\ \frac38\frac38\frac18$ | $16d\ .\bar{3}m\ \frac18\frac38\ {-\frac18}$ | **D** |
| | $.3m$–$.3m$ | 6-gon | $16d\ .\bar{3}m\ \frac38\frac38\frac58$ | $16c\ .\bar{3}m\ \frac18\frac38\frac38$ | $16d\ .\bar{3}m\ \frac38\frac58\frac38$ | $16c\ .\bar{3}m\ \frac38\frac38\frac18$ | **P** |
| | | | $16d\ .\bar{3}m\ \frac58\frac38\frac38$ | $16c\ .\bar{3}m\ \frac38\frac18\frac38$ | | | |
| $P\bar{4}3m$–$F\bar{4}3m(2a)$ | $.3m$–$.3m$ | 6-gon | $3d\ \bar{4}2.m\ 00\frac12$ | $3c\ \bar{4}2.m\ \frac12 0\frac12$ | $3d\ \bar{4}2.m\ \frac12 00$ | $3c\ \bar{4}2.m\ \frac12\frac12 0$ | **D** |
| | | | $3d\ \bar{4}2.m\ 0\frac12 0$ | $3c\ \bar{4}2.m\ 0\frac12\frac12$ | | | |
| $Ia\bar{3}d$–$I\bar{4}3d$ | $.\bar{3}.$–$.\bar{3}.$ | 12-gon | $24c\ 2.22\ \frac14\frac18 0$ | $16b\ .32\ {-\frac18}\frac18\ {-\frac38}$ | $24c\ 2.22\ {-\frac18}0\ {-\frac14}$ | $16b\ .32\ {-\frac18}\frac38\frac18$ | **S** |
| | | | $24c\ 2.22\ 0\frac14\frac18$ | $16b\ .32\ {-\frac38}\ {-\frac18}\frac18$ | $24c\ 2.22\ {-\frac14}\ {-\frac18}0$ | $16b\ .32\ \frac18\ {-\frac18}\frac38$ | |
| | | | $24c\ 2.22\ \frac18 0\frac14$ | $16b\ .32\ \frac18\ {-\frac38}\ {-\frac18}$ | $24c\ 2.22\ 0\ {-\frac14}\ {-\frac18}$ | $16b\ .32\ \frac38\frac18\ {-\frac18}$ | |
| $Ia\bar{3}d$–$Ia\bar{3}$ | $\bar{4}..$–$2..$ | 8-gon | $16b\ .32\ \frac18\frac18\frac18$ | $24c\ 2.22\ 0\frac14\frac18$ | $16b\ .32\ \frac38\frac58\frac18$ | $24c\ 2.22\ \frac38\frac12\frac14$ | **P** |
| | | | $16b\ .32\ \frac38\frac18\ {-\frac18}$ | $24c\ 2.22\ \frac12\frac14\ {-\frac18}$ | $16b\ .32\ \frac18\frac58\ {-\frac18}$ | $24c\ 2.22\ \frac18\frac12\ {-\frac14}$ | |
| $I4_132$–$P4_332$ | $..2$–$..2$ | 6-gon | $12c\ 2.22\ \frac18 0\frac14$ | $8b\ .32\ \frac18\ {-\frac38}\ {-\frac18}$ | $12d\ 2.22\ \frac14\ {-\frac38}0$ | $12c\ 2.22\ \frac14\frac18 0$ | **D** |
| | | | $8b\ .32\ \frac58\frac18\frac38$ | $12d\ 2.22\ \frac58 0\frac14$ | | | |
| | $.3.$–$.3.$ | 9-gon | $12c\ 2.22\ \frac18 0\frac14$ | $12d\ 2.22\ \frac58 0\frac14$ | $8b\ .32\ \frac58\frac18\frac38$ | $12c\ 2.22\ \frac14\frac18 0$ | **C(Y)** |
| | | | $12d\ 2.22\ \frac14\frac58 0$ | $8b\ .32\ \frac38\frac58\frac18$ | $12c\ 2.22\ 0\frac14\frac18$ | $12d\ 2.22\ 0\frac14\frac58$ | |
| | | | $8b\ .32\ \frac18\frac38\frac58$ | | | | |
| $P6_222$–$P6_122(2c)$ | $..2$–$..2$ | 8-gon | $3a\ 222\ 000$ | $3c\ 222\ \frac12 10$ | $3d\ 222\ \frac12 1\ {-\frac12}$ | $3b\ 222\ 11\ {-\frac12}$ | **HS1** |
| | | | $3a\ 222\ 11\ {-\frac13}$ | $3c\ 222\ 0\frac12\ {-\frac13}$ | $3d\ 222\ 0\frac12\frac16$ | $3b\ 222\ 00\frac16$ | |

Table 1. *Cont.*

| space-group pair $G$-$S$ | disc-like surface patch: site sym. | polygon | vertices | | | | minimal surface |
|---|---|---|---|---|---|---|---|
| $P6_222$–$P3_212$ | $..2$–$..2$ | 8-gon | $3a\ 222\ 000$ | $3c\ 222\ \frac{1}{2}00$ | $3d\ 222\ \frac{1}{2}0\frac{1}{2}$ | $3b\ 222\ 00\frac{1}{2}$ | **HS2** |
| | | | $3a\ 222\ 00\frac{1}{3}$ | $3c\ 222\ \frac{1}{2}\frac{1}{2}\frac{1}{3}$ | $3d\ 222\ \frac{1}{2}\frac{1}{2}-\frac{1}{6}$ | $3b\ 222\ 00-\frac{1}{6}$ | |
| $P4_2/mcm$– | $m.m2$–$mm2.$ | 6-gon | $4e\ 222.\ \frac{1}{2}0\frac{1}{4}$ | $2d\ \bar{4}2m\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $4e\ 222.\ 0\frac{1}{2}\frac{1}{4}$ | $4e\ 222.\ 0\frac{1}{2}-\frac{1}{4}$ | **CLP** |
| $P4_2/mmc(v)$ | | | $2d\ \bar{4}2m\ \frac{1}{2}\frac{1}{2}-\frac{1}{4}$ | $4e\ 222.\ \frac{1}{2}0-\frac{1}{4}$ | | | |
| $P4_2/nnm$– | $..m$–$.m.$ | 5-gon | $2b\ \bar{4}2m\ 00\frac{1}{2}$ | $4c\ 222.\ \frac{1}{2}0\frac{1}{2}$ | $4d\ 2.22\ \frac{1}{2}0\frac{3}{4}$ | $4d\ 2.22\ 0\frac{1}{2}\frac{3}{4}$ | **tD** |
| $I4_1/amd(v,2c)$ | | | $4c\ 222.\ 0\frac{1}{2}\frac{1}{2}$ | | | | |
| | $2.mm$–$2mm.$ | 10-gon | $2b\ \bar{4}2m\ 00\frac{1}{2}$ | $4c\ 222.\ \frac{1}{2}0\frac{1}{2}$ | $4d\ 2.22\ \frac{1}{2}0\frac{1}{4}$ | $4d\ 2.22\ 1\frac{1}{2}\frac{1}{4}$ | **tD** |
| | | | $4c\ 222.\ 1\frac{1}{2}\frac{1}{2}$ | $2b\ \bar{4}2m\ 11\frac{1}{2}$ | $4c\ 222.\ \frac{1}{2}1\frac{1}{2}$ | $4d\ 2.22\ \frac{1}{2}1\frac{1}{4}$ | |
| | | | $4d\ 2.22\ 0\frac{1}{2}\frac{1}{4}$ | $4c\ 222.\ 0\frac{1}{2}\frac{1}{2}$ | | | |
| $P4_222$–$P4_122(2c)$ | $1$–$1$ | 6-gon | $2a\ 222.\ 000$ | $2d\ 222.\ \frac{1}{2}00$ | $2c\ 222.\ \frac{1}{2}0\frac{1}{2}$ | $2b\ 222.\ \frac{1}{2}\frac{1}{2}\frac{1}{2}$ | **tD** |
| | | | $2f\ 2.22\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $2e\ 2.22\ 00\frac{1}{4}$ | | | |
| | $..2$–$..2$ | 12-gon | $2e\ 2.22\ 00\frac{1}{4}$ | $2f\ 2.22\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $2b\ 222.\ \frac{1}{2}\frac{1}{2}\frac{1}{2}$ | $2c\ 222.\ \frac{1}{2}0\frac{1}{2}$ | **tD** |
| | | | $2d\ 222.\ \frac{1}{2}01$ | $2a\ 222.\ 001$ | $2e\ 2.22\ 00\frac{5}{4}$ | $2f\ 2.22\ \frac{1}{2}\frac{1}{2}\frac{5}{4}$ | |
| | | | $2b\ 222.\ \frac{1}{2}\frac{1}{2}1$ | $2c\ 222.\ 0\frac{1}{2}1$ | $2d\ 222.\ 0\frac{1}{2}\frac{1}{2}$ | $2a\ 222.\ 00\frac{1}{2}$ | |
| $P\bar{4}2m$–$I\bar{4}m2(v,2c)$ | $..m$–$.m.$ | 6-gon | $1c\ \bar{4}2m\ 00\frac{1}{2}$ | $2f\ 222.\ \frac{1}{2}0\frac{1}{2}$ | $2e\ 222.\ \frac{1}{2}00$ | $1d\ \bar{4}2m\ \frac{1}{2}\frac{1}{2}0$ | **tD** |
| | | | $2e\ 222.\ 0\frac{1}{2}0$ | $2f\ 222.\ 0\frac{1}{2}\frac{1}{2}$ | | | |
| $P\bar{4}n2$–$I\bar{4}2d(v,2c)$ | $\bar{4}..$–$2..$ | 8-gon | $2d\ 2.22\ \frac{1}{2}0\frac{1}{4}$ | $2c\ 2.22\ \frac{1}{2}0\frac{3}{4}$ | $2d\ 2.22\ 0\frac{1}{2}\frac{3}{4}$ | $2c\ 2.22\ 0\frac{1}{2}\frac{1}{4}$ | **tD** |
| | | | $2d\ 2.22\ -\frac{1}{2}0\frac{1}{4}$ | $2c\ 2.22\ -\frac{1}{2}0\frac{3}{4}$ | $2d\ 2.22\ 0-\frac{1}{2}\frac{3}{4}$ | $2c\ 2.22\ 0-\frac{1}{2}\frac{1}{4}$ | |
| $Pccm$– | $..m$–$..m$ | 6-gon | $2f\ 222\ \frac{1}{2}0\frac{1}{4}$ | $2h\ 222\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $2g\ 222\ 0\frac{1}{2}\frac{1}{4}$ | $2g\ 222\ 0\frac{1}{2}-\frac{1}{4}$ | **oCLP** |
| $Cccm(2a,2b)$ | | | $2h\ 222\ \frac{1}{2}\frac{1}{2}-\frac{1}{4}$ | $2f\ 222\ \frac{1}{2}0-\frac{1}{4}$ | | | |
| $Pnnn$– | $\bar{1}-1$ | 6-gon | $2c\ 222\ 00\frac{1}{2}$ | $2d\ 222\ \frac{1}{2}0\frac{1}{2}$ | $2b\ 222\ \frac{1}{2}01$ | $2c\ 222\ \frac{1}{2}\frac{1}{2}1$ | **oDa** |
| $Fddd(2a,2b,2c)$ | | | $2d\ 222\ 0\frac{1}{2}1$ | $2b\ 222\ 0\frac{1}{2}\frac{1}{2}$ | | | |
| $Cmma$– | $mm2$–$mm2$ | 8-gon | $4b\ 222\ \frac{1}{4}0\frac{1}{2}$ | $4b\ 222\ \frac{3}{4}0\frac{1}{2}$ | $4a\ 222\ \frac{3}{4}00$ | $4a\ 222\ \frac{3}{4}\frac{1}{2}0$ | **oDb** |
| $Imma(2c)$ | | | $4b\ 222\ \frac{3}{4}\frac{1}{2}\frac{1}{2}$ | $4b\ 222\ \frac{1}{4}\frac{1}{2}\frac{1}{2}$ | $4a\ 222\ \frac{1}{4}\frac{1}{2}0$ | $4a\ 222\ \frac{1}{4}00$ | |

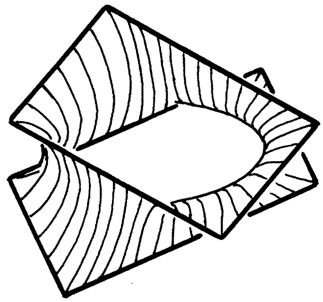

Figure 2. 4-catenoid, surface patch of an HS3 surface.

example of a minimal surface generated from such a surface patch is the H surface of Schwarz (1890). Such a spanning exists in general as a solution of the anular Plateau problem, but only as long as the distance between the polygons is not too large compared to their size. This restriction corresponds to a limit for the axial ratio(s). For example, within the permissible range of the axial ratio there exists a whole family of H surfaces, rather than a unique 'H surface'.

By analogy to table 1, all catenoid-like surface patches (Koch & Fischer 1988) and the corresponding infinite minimal surfaces are described in table 2. By contrast to disc-like surface patches, catenoid-like surface patches have necessarily the same site symmetry with respect to $G$ and to $S$.

In order to calculate the Euler characteristic of such a surface, catenoids cannot be used directly for a tiling, but one disc-like tile may be produced from each catenoid by cutting it up between two vertices. In this way one additional edge per catenoid is generated. If $k$ is the number of catenoids per primitive unit cell of $S$, the Euler characteristic is given by (Fischer & Koch 1989*c*)

$$\chi = -2k. \tag{2.3}$$

### (*d*) *Branched catenoids*

If a line configuration can be decomposed into parallel plane nets of the two different kinds, the corresponding polygons have different areas. The area of the larger polygon is an integer multiple, say $n$–fold, of the area of the smaller polygon. A ring-like surface patch spanning the gap between a small and a large polygon would result in a minimal surface with self-intersections. In all three cases of such alternating nets it is possible, however, to stretch a surface patch from one polygon of the wider net to the union of $n$ polygons of the finer net (cf. figure 3). These $n$ polygons share one vertex lying directly above or below the centre of the larger polygon. The resulting surface patch resembles a catenoid at one end, but it branches into $n$ openings at the other end (Fischer & Koch 1989*a*).

The existence of branched catenoids seems to be evident and is easily confirmed by soap-film experiments, but a rigorous mathematical proof is not trivial (Nitsche 1990). Again, branched catenoids can exist only if an upper limit for the distance of neighbouring nets is not exceeded. Table 3 gives details of these surface patches and the corresponding infinite minimal surfaces.

If a primitive unit cell of $S$ contains $k$ catenoids, each of which has $b$ branches at one end, then the Euler characteristic is (Fischer & Koch 1989*c*)

$$\chi = -(1 + b)k. \tag{2.4}$$

Table 2. *Catenoid-like surface patches generating 3-periodic minimal surfaces without self-intersections*

| space-group pair $G$-$S$ | catenoid-like surface patch: site sym. | polygons | vertices | | | | minimal surface |
|---|---|---|---|---|---|---|---|
| $P6_3/mmc$– | $\bar{6}m2$ | 3-gon | $2a\ \bar{3}m.\ 00\frac{1}{2}$ | $2a\ \bar{3}m.\ 10\frac{1}{2}$ | $2a\ \bar{3}m.\ 11\frac{1}{2}$ | | **H** |
| $P\bar{6}m2$ | | 3-gon | $2a\ \bar{3}m.\ 000$ | $2a\ \bar{3}m.\ 100$ | $2a\ \bar{3}m.\ 110$ | | |
| $P6/mcc$– | $m..$ | 3-gon | $2a\ 622\ 00\frac{1}{4}$ | $6f\ 222\ \frac{1}{2}0\frac{1}{4}$ | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\frac{1}{4}$ | | **R3** |
| $P6/m$ | | 3-gon | $2a\ 622\ 00\ {-\frac{1}{4}}$ | $6f\ 222\ \frac{1}{2}0\ {-\frac{1}{4}}$ | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\ {-\frac{1}{4}}$ | | |
| $P6_222$– | $222$ | 4-gon | $3b\ 222\ 00\frac{1}{2}$ | $3d\ 222\ \frac{1}{2}0\frac{1}{2}$ | $3b\ 222\ 11\frac{1}{2}$ | $3d\ 222\ \frac{1}{2}1\frac{1}{2}$ | **HS3** |
| $P6_422(2c)$ | | 4-gon | $3b\ 222\ 00\frac{1}{6}$ | $3d\ 222\ 1\frac{1}{2}\frac{1}{6}$ | $3b\ 222\ 11\frac{1}{6}$ | $3d\ 222\ 0\frac{1}{2}\frac{1}{6}$ | |
| $R\bar{3}m$– | $\bar{3}m$ | 3-gon | $3b\ \bar{3}m\ 00\frac{1}{2}$ | $3b\ \bar{3}m\ 10\frac{1}{2}$ | $3b\ \bar{3}m\ 11\frac{1}{2}$ | | **rPD** |
| $R\bar{3}m(2c)$ | | 3-gon | $3b\ \bar{3}m\ \frac{1}{3}\ {-\frac{1}{3}}\frac{1}{6}$ | $3b\ \bar{3}m\ \frac{4}{3}\frac{2}{3}\frac{1}{6}$ | $3b\ \bar{3}m\ \frac{1}{3}\frac{2}{3}\frac{1}{6}$ | | |
| $I4/mmm$– | $4/mmm$ | 4-gon | $4d\ \bar{4}m2\ \frac{1}{2}0\frac{1}{4}$ | $4d\ \bar{4}m2\ 1\frac{1}{2}\frac{1}{4}$ | $4d\ \bar{4}m2\ \frac{1}{2}1\frac{1}{4}$ | $4d\ \bar{4}m2\ 0\frac{1}{2}\frac{1}{4}$ | **tP** |
| $P4/mmm$ | | 4-gon | $4d\ \bar{4}m2\ \frac{1}{2}0\ {-\frac{1}{4}}$ | $4d\ \bar{4}m2\ 1\frac{1}{2}\ {-\frac{1}{4}}$ | $4d\ \bar{4}m2\ \frac{1}{2}1\ {-\frac{1}{4}}$ | $4d\ \bar{4}m2\ 0\frac{1}{2}\ {-\frac{1}{4}}$ | |
| $I4/mcm$– | $m.m2$ | 3-gon | $4a\ 422\ 00\frac{1}{4}$ | $4b\ \bar{4}2m\ \frac{1}{2}0\frac{1}{4}$ | $4a\ 422\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | | **R2** |
| $P4/mbm$ | | 3-gon | $4a\ 422\ 00\ {-\frac{1}{4}}$ | $4b\ \bar{4}2m\ \frac{1}{2}0\ {-\frac{1}{4}}$ | $4a\ 422\ \frac{1}{2}\frac{1}{2}\ {-\frac{1}{4}}$ | | |
| $Fmmm$– | $mmm$ | 4-gon | $8f\ 222\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$ | $8f\ 222\ \frac{3}{4}\frac{1}{4}\frac{1}{4}$ | $8f\ 222\ \frac{3}{4}\frac{3}{4}\frac{1}{4}$ | $8f\ 222\ \frac{1}{4}\frac{3}{4}\frac{1}{4}$ | **oPb** |
| $Cmmm$ | | 4-gon | $8f\ 222\ \frac{1}{4}\frac{1}{4}\ {-\frac{1}{4}}$ | $8f\ 222\ \frac{3}{4}\frac{1}{4}\ {-\frac{1}{4}}$ | $8f\ 222\ \frac{3}{4}\frac{3}{4}\ {-\frac{1}{4}}$ | $8f\ 222\ \frac{1}{4}\frac{3}{4}\ {-\frac{1}{4}}$ | |

Table 3. *Branched catenoids generating 3-periodic minimal surfaces without self-intersections*

| space-group pair $G$-$S$ | branched catenoid: site sym. | polygons | vertices | | | | minimal surface |
|---|---|---|---|---|---|---|---|
| $P6_322$–$P6_3$ | 3.. | 3-gon | $2a$ 32. 000 | $2a$ 32. 100 | $2a$ 32. 110 | | **BC1** |
| | | 9-gon | $2d$ 3.2 $\frac{2}{3}\frac{1}{3}\frac{1}{4}$ | $2b$ 3.2 $00\frac{1}{4}$ | $2c$ 3.2 $\frac{1}{3}\,-\frac{1}{3}\frac{1}{4}$ | $2d$ 3.2 $\frac{2}{3}\frac{1}{3}\frac{1}{4}$ | |
| | | | $2b$ 3.2 $10\frac{1}{4}$ | $2c$ 3.2 $\frac{4}{3}\frac{2}{3}\frac{1}{4}$ | $2d$ 3.2 $\frac{2}{3}\frac{1}{3}\frac{1}{4}$ | $2b$ 3.2 $11\frac{1}{4}$ | |
| | | | $2c$ 3.2 $\frac{1}{3}\frac{2}{3}\frac{1}{4}$ | | | | |
| $P4_2/nnm$–$P4_2nm$ | 2.$mm$ | 4-gon | $4d$ 2.22 $\frac{1}{2}0\frac{1}{4}$ | $4d$ 2.22 $0\frac{1}{2}\frac{1}{4}$ | $4d$ 2.22 $-\frac{1}{2}0\frac{1}{4}$ | $4d$ 2.22 $0\,-\frac{1}{2}\frac{1}{4}$ | **BC2** |
| | | 8-gon | $2a$ $\bar{4}2m$ 000 | $4c$ 222. $\frac{1}{2}00$ | $2b$ $\bar{4}2m$ $\frac{1}{2}\frac{1}{2}0$ | $4c$ 222. $0\frac{1}{2}0$ | |
| | | | $2a$ $\bar{4}2m$ 000 | $4c$ 222. $-\frac{1}{2}00$ | $2b$ $\bar{4}2m$ $-\frac{1}{2}\,-\frac{1}{2}0$ | $4c$ 222. $0\,-\frac{1}{2}0$ | |
| $I422$–$I4$ | 4.. | 4-gon | $4d$ 2.22 $\frac{1}{2}0\frac{1}{4}$ | $4d$ 2.22 $0\frac{1}{2}\frac{1}{4}$ | $4d$ 2.22 $-\frac{1}{2}0\frac{1}{4}$ | $4d$ 2.22 $0\,-\frac{1}{2}\frac{1}{4}$ | **BC3** |
| | | 12-gon | $2a$ 422 000 | $2b$ 422 $\frac{1}{2}\frac{1}{2}0$ | $4c$ 222. $0\frac{1}{2}0$ | $2a$ 422 000 | |
| | | | $2b$ 422 $-\frac{1}{2}\frac{1}{2}0$ | $4c$ 222. $-\frac{1}{2}00$ | $2a$ 422 000 | $2b$ 422 $-\frac{1}{2}\,-\frac{1}{2}0$ | |
| | | | $4c$ 222. $0\,-\frac{1}{2}0$ | $2a$ 422 000 | $2b$ 422 $\frac{1}{2}\,-\frac{1}{2}0$ | $4c$ 222. $\frac{1}{2}00$ | |

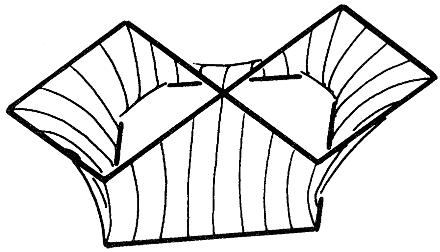

Figure 3. Two-fold branched catenoid, surface patch of a BC2 surface.

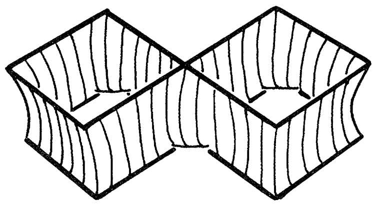

Figure 4. Double catenoid, surface patch of an MC5 surface.

### (*e*) *Multiple catenoids*

Multiple catenoids are also ring-like surface patches, but they are branched at both ends. Whereas a branched catenoid is stretched between a convex polygon and a concave polygon with one point of self-contact, a multiple catenoid connects two congruent concave polygons, the self-contacting points of which must lie directly above each other. It may be visualized as resulting from the fusion of up to six neighbouring catenoids (cf. figure 4). Such surface patches can only be formed between congruent plane nets stacked directly one upon another, provided the distance between the nets is sufficiently small.

Surfaces built up from multiple catenoids are sumarized in table 4. Most of these have been derived independently by Karcher (1989) and Koch & Fischer (1989*a*). By contrast to catenoids and branched catenoids, the two bounding polygons of multiple catenoids must always lie directly upon each other with a distance of one half of the corresponding lattice constant. Therefore, table 4 describes the vertices of only one bounding polygon of each multiple catenoid. For the corresponding infinite surfaces the Euler characteristic is

$$\chi = -2mk, \tag{2.5}$$

where $k$ is the number of $m$-fold catenoids per primitive unit cell of $S$ (Fischer & Koch 1989*c*).

### (*f*) *Infinite strips*

If the fusion of two catenoids is repeated infinitely many times along a straight line, a channel bounded by two infinite strips results. The first boundary of such a strip may either be a straight line, a zigzag or a meander line, whereas the second boundary must be a zigzag or a meander line. Both boundaries of a strip must extend in the same direction and their middle lines must lie directly above each other (cf. figure 5). Strip-like surface patches are only compatible with rectangular nets of 2-fold axes or with line configurations which can be decomposed into congruent rectangular

Table 4. *Multiple catenoids generating 3-periodic minimal surfaces without self-intersections*

| space-group pair $G$-$S$ | multiple catenoid: site sym. | polygons | vertices | | | | minimal surface |
|---|---|---|---|---|---|---|---|
| $P6_3/mcm$–$P\bar{6}2m$ | $\bar{6}2m$ | 9-gons | $2b\ \bar{3}.m\ 000$ | $4d\ 3.2\ \frac{2}{3}\frac{1}{3}0$ | $4d\ 3.2\ \frac{1}{3}\frac{2}{3}0$ | $2b\ \bar{3}.m\ 000$ | **MC1** |
| | | | $4d\ 3.2\ -\frac{1}{3}\frac{1}{3}0$ | $4d\ 3.2\ -\frac{2}{3}-\frac{1}{3}0$ | $2b\ \bar{3}.m\ 000$ | $4d\ 3.2\ -\frac{1}{3}-\frac{2}{3}0$ | |
| | | | $4d\ 3.2\ \frac{1}{3}-\frac{1}{3}0$ | | | | |
| $P6/mcc$–$P6/m$ | $2/m..$ | 6-gons | $6f\ 222\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $4c\ 3.2\ \frac{1}{3}\frac{2}{3}\frac{1}{4}$ | $2a\ 622\ 00\frac{1}{4}$ | $6f\ 222\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | **MC2** |
| | | | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\frac{1}{/}4$ | $2a\ 622\ 11\frac{1}{4}$ | | | |
| $P6/mcc$–$P6/m$ | $\bar{6}..$ | 9-gons | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\frac{1}{4}$ | $2a\ 622\ 00\frac{1}{4}$ | $6f\ 222\ \frac{1}{2}0\frac{1}{4}$ | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\frac{1}{4}$ | **MC3** |
| | | | $2a\ 622\ 10\frac{1}{4}$ | $6f\ 222\ 1\frac{1}{2}\frac{1}{4}$ | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\frac{1}{4}$ | $2a\ 622\ 11\frac{1}{4}$ | |
| | | | $6f\ 222\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | | | | |
| $P6/mcc$–$P6/m$ | $6/m..$ | 18-gons | $2a\ 622\ 00\frac{1}{4}$ | $6f\ 222\ \frac{1}{2}0\frac{1}{4}$ | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\frac{1}{4}$ | $2a\ 622\ 00\frac{1}{4}$ | **MC4** |
| | | | $6f\ 222\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $4c\ 3.2\ \frac{1}{3}\frac{2}{3}\frac{1}{4}$ | $2a\ 622\ 00\frac{1}{4}$ | $6f\ 222\ 0\frac{1}{2}\frac{1}{4}$ | |
| | | | $4c\ 3.2\ -\frac{1}{3}\frac{1}{3}\frac{1}{4}$ | $2a\ 622\ 00\frac{1}{4}$ | $6f\ 222\ -\frac{1}{2}0\frac{1}{4}$ | $4c\ 3.2\ -\frac{2}{3}-\frac{1}{3}\frac{1}{4}$ | |
| | | | $2a\ 622\ 00\frac{1}{4}$ | $6f\ 222\ -\frac{1}{2}-\frac{1}{2}\frac{1}{4}$ | $4c\ 3.2\ -\frac{1}{3}-\frac{2}{3}\frac{1}{4}$ | $2a\ 622\ 00\frac{1}{4}$ | |
| | | | $6f\ 222\ 0-\frac{1}{2}\frac{1}{4}$ | $4c\ 3.2\ \frac{1}{3}-\frac{1}{3}\frac{1}{4}$ | | | |
| $P4_2/mcm$–$Cmmm(v)$ | $m.mm$ | 8-gons | $2b\ \bar{4}2m\ 00\frac{1}{4}$ | $4e\ 222.\ \frac{1}{2}0\frac{1}{4}$ | $2d\ \bar{4}2m\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $4e\ 222.\ 0\frac{1}{2}\frac{1}{4}$ | **MC5** |
| | | | $2b\ \bar{4}2m\ 00\frac{1}{4}$ | $4e\ 222.\ -\frac{1}{2}0\frac{1}{4}$ | $2d\ \bar{4}2m\ -\frac{1}{2}-\frac{1}{2}\frac{1}{4}$ | $4e\ 222.\ 0-\frac{1}{2}\frac{1}{4}$ | |
| $I4/mcm$–$P4/mbm$ | $m.mm$ | 6-gons | $4b\ \bar{4}2m\ \frac{1}{2}0\frac{1}{4}$ | $4a\ 422\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $4a\ 422\ 00\frac{1}{4}$ | $4b\ \bar{4}2m\ \frac{1}{2}0\frac{1}{4}$ | **MC6** |
| | | | $4a\ 422\ \frac{1}{2}-\frac{1}{2}\frac{1}{4}$ | $4a\ 422\ 10\frac{1}{4}$ | | | |
| $P4/mcc$–$P4/m$ | $4/m..$ | 12-gons | $2a\ 422\ 00\frac{1}{4}$ | $4f\ 222.\ \frac{1}{2}0\frac{1}{4}$ | $2c\ 422\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $2a\ 422\ 00\frac{1}{4}$ | **MC7** |
| | | | $4f\ 222.\ 0\frac{1}{2}\frac{1}{4}$ | $2c\ 422\ -\frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $2a\ 422\ 00\frac{1}{4}$ | $4f\ 222.\ -\frac{1}{2}0\frac{1}{4}$ | |
| | | | $2c\ 422\ -\frac{1}{2}-\frac{1}{2}\frac{1}{4}$ | $2a\ 422\ 00\frac{1}{4}$ | $4f\ 222.\ 0-\frac{1}{2}\frac{1}{4}$ | $2c\ 422\ \frac{1}{2}-\frac{1}{2}\frac{1}{4}$ | |
| $Pccm$–$P2/m$ | $..2/m$ | 8-gons | $2e\ 222\ 00\frac{1}{4}$ | $2f\ 222\ \frac{1}{2}0\frac{1}{4}$ | $2h\ 222\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $2g\ 222\ 0\frac{1}{2}\frac{1}{4}$ | **oMC5** |
| | | | $2e\ 222\ 00\frac{1}{4}$ | $2f\ 222\ -\frac{1}{2}0\frac{1}{4}$ | $2h\ 222\ -\frac{1}{2}-\frac{1}{2}\frac{1}{4}$ | $2g\ 222\ 0-\frac{1}{2}\frac{1}{4}$ | |

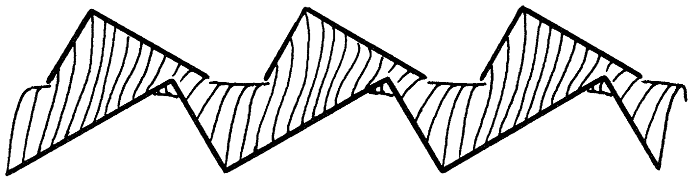

Figure 5. Infinite strip, surface patch of an ST1 surface.

nets and isolated 2-fold axes parallel to these nets. However, most such strips may be subdivided by additional 2-fold axes into disc-like surface patches and, therefore, refer to previously known families of minimal surfaces.

The only two strip-like surface patches giving rise to new families (Fischer & Koch 1989*b*) are described in table 5. By contrast to catenoids, branched catenoids and multiple catenoids, for which the distance between the parallel nets must not exceed a certain limit, no such restriction applies to infinite strips: the larger the distance between the nets, the more the surface patch resembles a plane with two undulating rims.

If $k$ is the number of original catenoids per primitive unit cell of $S$ which have been united to form the infinite rows, then (Fischer & Koch 1989*c*)

$$\chi = -2k. \qquad (2.6)$$

### (*g*) *Catenoids with spout-like attachments*

By analogy to the C(P) surface of Neovius (1883), Schoen (1970) derived a new family of minimal surfaces. He designated it as C(H) because such a surface spans the same line configuration as the H surface. In his derivation Schoen used kaleidoscopic cells, a procedure which cannot be generalized, but the resulting surface patch may be. The relationship between an H surface and a C(H) surface may be visualized as follows. The line configuration spanning an H or a C(H) surface consists of parallel triangular nets stacked directly upon each other. In an H surface, catenoid-like surface patches are located in every second trigonal prism between two adjacent nets. The corresponding C(H) surface results from an H surface by pulling three spouts out of each catenoid and uniting every three of them into a three-armed handle inside those trigonal prisms which are left empty by the H surface. The surface patch of an C(H) surface may therefore be considered as a catenoid with three attached spouts. In contrast to all other surface patches described so far, it is not entirely bounded by straight lines, but the spouts end on mirror planes, i.e. in plane lines of curvature. A similar procedure has been applied to other catenoid-like surface patches by Koch & Fischer (1989*b*) resulting in five new families of spanning minimal surfaces (cf. figure 6).

Table 6 contains information on these families: the symmetry of an infinite surface in column 1, the symbol of the family of catenoid surfaces (cf. table 2) which has been used for the derivation in column 2, the symmetry (point group) of a single catenoid with spout-like attachments in column 3, the symmetry (layer group or rod group) of a smallest part of the infinite surface entirely bounded by straight lines in column 4 (for the symbols of layer and rod groups, cf. Bohm & Dornberger-Schiff 1967), and the symbol of the surface family in column 5.

Soap film experiments on the PT surface confirm its existence and seem to show

Table 5. *Strip-like surface patches generating 3-periodic minimal surfaces without self-intersections*

| space-group pair $G$-$S$ | infinite strip: dir. | symmetry | border lines | vertices | | | | | minimal surface |
|---|---|---|---|---|---|---|---|---|---|
| $P6_222$– | ⟨010⟩ | $P(12)1$– | zigzag | ... | $3b$ 222 $00\frac{1}{6}$ | $3d$ 222 $1\frac{1}{2}\frac{1}{6}$ | $3b$ 222 $10\frac{1}{6}$ | ... | **ST1** |
| $P6_422(2c)$ | | $P(12)1$ | zigzag | ... | $3b$ 222 $00-\frac{1}{6}$ | $3d$ 222 $\frac{1}{2}\frac{1}{2}-\frac{1}{6}$ | $3b$ 222 $10-\frac{1}{6}$ | ... | |
| $P4_2/nbc$– | ⟨100⟩ | $P(1c)1$– | meander | ... $4a$ 222. $0\frac{1}{2}\frac{1}{4}$ | $4b$ 222. $00\frac{1}{4}$ | $4a$ 222. $\frac{1}{2}0\frac{1}{4}$ | $4b$ 222. $\frac{1}{2}\frac{1}{2}\frac{1}{4}$ | $4a$ 222. $1\frac{1}{2}\frac{1}{4}$ ... | **ST2** |
| $P4_2/n$ | | $P(11)1$ | zigzag | ... | $4c$ 2.22 $0\frac{1}{2}0$ | $4c$ 2.22 $\frac{1}{2}00$ | $4c$ 2.22 $1\frac{1}{2}0$ | ... | |

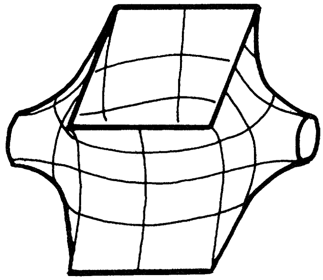

Figure 6. 4-catenoid with two spouts, surface patch of a PT surface.

Table 6. *Catenoids with spout-like attachments generating 3-periodic minimal surfaces without self-intersections*

| space-group pair $G$–$S$ | original catenoid surf. | sym. of surface patch: finite | sym. of surface patch: infinite | minimal surface |
|---|---|---|---|---|
| $P6_3/mmc$–$P\bar{6}m2$ | H | $\bar{6}m2$ | $P(\bar{6})m2$ | **C(H)** |
| $I4/mcm$–$P4/mbm$ | R2 | $m.m2$ | $P(4/m)bm$ | **C(R2)** |
| $P6/mcc$–$P6/m$ | R3 | $m..$ | $P(6/m)11$ | **C(R3)** |
| $I4/mmm$–$P4/mmm$ | tP | $4/mmm$ | $P(4/m)mm$ | **tC(P)** |
| $Fmmm$–$Cmmm$ | oPb | $mmm$ | $Cmm(m)$ | **oC(P)** |
| | | $mmm$ | $Cm(mm)$ | **PT** |

that the handles stabilize the surface towards larger separations of the nets in comparison to catenoid surfaces. However, a rigorous proof is lacking. Attaching $s$ spouts to each catenoid results in multiplying $\chi$ by a factor of $s$ in (2.3), giving (Fischer & Koch 1989*c*)

$$\chi = -2sk. \tag{2.7}$$

### (*h*) *Discs with spout-like attachments*

In a C(P) surface there exist pairs of disc-like surface patches which share every second vertex and show a comparatively small distance between their middle regions, i.e. between their centring flat points. Accordingly, it should be possible to connect such two discs by a handle (Karcher 1989; Karcher & Polthier 1990). The resulting surface patch is a disc with one spout attached at its middle, again only partly bounded by straight lines (cf. figure 7). A similar situation occurs for C(D) and C(Y) surfaces (Koch & Fischer 1993*b*).

Table 7 gives information on three families of such surfaces: the symmetry of an infinite surface, the symbol of the family of surfaces with disc-like surface patches used for the derivation (cf. table 1), the point-group symmetry of a single disc with a spout-like attachment and of a double surface patch stretched between two polygons which share some of their vertices, and the symbol of the new surface family. Inserting the handles modifies (2.2) to

$$\chi = -\tfrac{1}{2}fe_p + \sum_i v_i. \tag{2.8}$$

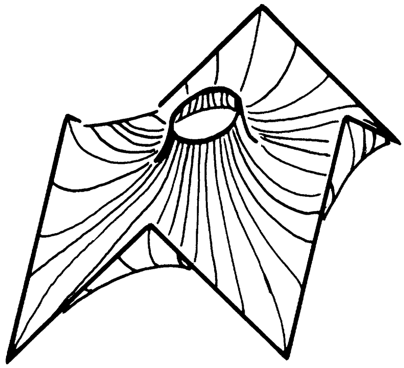

Figure 7. Spanned 8-gon with one spout, surface patch of a C(P)/H surface.

Table 7. *Discs with spout-like attachments generating 3-periodic minimal surfaces without self-intersections*

| space-group pair $G$–$S$ | original surface | sym. of surface patch single | sym. of surface patch double | minimal surface |
|---|---|---|---|---|
| $Im\bar{3}m$–$Pm\bar{3}m$ | C(P) | $4m.m$ | $4/mm.m$ | **C(P)/H** |
| $Pn\bar{3}m$–$Fd\bar{3}m(2a)$ | C(D) | $.3m$ | $.\bar{3}m$ | **C(D)/H** |
| $I4_132$–$P4_332$ | C(Y) | .3. | .32 | **C(Y)/H** |

## (*i*) *Other surface patches*

Catenoids and discs with spout-like attachments are examples of surface patches not entirely bounded by straight lines. There are other spanning minimal surfaces which also cannot be subdivided into finite surface patches with all boundaries formed by 2-fold axes.

### (i) *The family of VAL surfaces*

By constructing an appropriate Weierstrass function, Fogden & Hyde (1992*b*, cf. also Fogden 1993) derived a new family of minimal surfaces, called VAL surfaces, which are related to the PT family (cf. table 6). In a PT surface a catenoid is continued across its two spouts and this procedure is repeated *ad infinitum.* The resulting infinite surface patch looks like a perforated tube. Each hole in the tube is formed by a rectangle of 2-fold axes and the holes are located pairwise at the opposite sides of the tube. A VAL surface likewise may be dissected into perforated tubes spanned between ribbons of rectangles, but here the holes on the two sides alternate. Accordingly, the symmetry of a VAL surface ($Cmma$–$Cmma(2c)$) differs from that of the corresponding PT surface ($Fmmm$–$Cmmm$), although both span the same line configuration. A simple finite surface patch of a VAL surface is a disc with eight vertices. Four of its edges are straight lines and four are plane lines of curvature. Table 8 shows for each pair of vertices whether the corresponding edge refers to a 2-fold axis or lies in a mirror plane. An infinite VAL surface is generated from such a surface patch by means of the two reflection principles of Schwarz (1890).

(ii) *Surfaces spanning isolated 2-fold axes*

In the case of isolated 2-fold axes running in three linearly independent directions, surfaces have been constructed which contain these axes and seem to be minimal surfaces. Such a situation occurs for the group-subgroup pairs $Ia\bar{3}$–$Pa\bar{3}$ and $Ibca$–$Pbca$. If a cube with $0 \leqslant x, y, z \leqslant \frac{1}{2}$ (referred to the conventional setting of $Ia\bar{3}$) is considered, then six skew 2-fold axes bisect its faces and run parallel to its edges. Inside such a cube a disc-like surface patch with site symmetry $\bar{3}$ at $\frac{1}{4}\frac{1}{4}\frac{1}{4}$ span the six 2-fold axes. In addition, the patch has six curved borderlines alternating with the straight boundaries. If such a surface patch is continued across the 2-fold axes, the resulting infinite surface acquires triangular holes (with curved edges) arranged in pairs around the Wyckoff position 16*a* $.\bar{3}.$ at 000. It is possible, however, to close up the two holes of each pair, either by a ring-like patch giving rise to the family of $^{\pm}$Y surfaces (Fischer & Koch 1987, 1989*c*), or by two discs resulting in the family of C($^{\pm}$Y) surfaces (Koch & Fischer 1988; Fischer & Koch 1989*c*).

Instead of generating an infinite surface from two different surface patches, one may use larger disc-like surface patches of only one kind, namely skew 18-gons (cf. table 8) with alternately one straight edge and two curved edges.

$^{\pm}$Y surfaces as well as C($^{\pm}$Y) surfaces can be orthorhombically distorted resulting in o$^{\pm}$Y surfaces and in oC($^{\pm}$Y) surfaces (Koch & Fischer 1988; Fischer & Koch 1990). As all surface patches described in table 8 are disc-like, (2.2) may be used to calculate the Euler characteristics of the corresponding infinite surfaces.

### (*j*) *Genera*

Frequently, instead of the Euler characteristic, another entity, the *genus*, $g$, is used for the topological characterization of a surface in $R^3$. For a finite or infinite orientable surface the genus is defined as

$$g = 1 - \tfrac{1}{2}(\chi + r), \tag{2.9}$$

where $r$ is the number of boundaries of the surface. In the case of a surface without boundaries, e.g. a 3-periodic minimal surface, this formula reduces to

$$g = 1 - \tfrac{1}{2}\chi. \tag{2.10}$$

The advantage of using the genus instead of the Euler characteristic is its graphic meaning. A finite surface of genus $g$ is the topological equivalent of a sphere with $g$ handles, i.e. a torus has genus 1, a pretzel genus 2, etc. In other words, $g$ is the number of cuts that have to be applied to a connected surface to make it simply connected. A modified approach has been introduced for 3-periodic minimal surfaces (Schoen 1970) of counting only the number of handles per unit cell. For this, the surface is thought to be embedded in a (flat) 3-torus $T^3$, i.e. opposite faces of a primitive unit cell are identified in order to get rid of all translations (for a popular treatment of such embeddings, cf. Weeks 1985). In other words, an object moving within one labyrinth and leaving the unit cell across one face will re-enter it through the translationally equivalent opening on the opposite face. It is obvious that thereby the unit cell has to refer to the subgroup $S$, because otherwise the moving object might pass into the other labyrinth. The genus of the finite surface constructed in this way may either be calculated from (2.10) or by counting its handles or the number of cuts. As it is difficult to visualize a 3-periodic surface, Schoen (1970) proposed to represent each labyrinth by a graph (skeletal graph in his terminology) revealing the connectedness of the surface and the mutual interpenetration of its

Table 8. *Other disc-like surface patches generating 3-periodic minimal surfaces without self-intersections*
(These surface patches are only partly bounded by twofold axes [2].
The other boundaries are either plane lines of curvature [*m*] or general curved lines [1].)

| space-group pair $G$-$S$ | other disc-like surface patch: site sym. | polygon | vertices | | | | | | minimal surface |
|---|---|---|---|---|---|---|---|---|---|
| $Cmma$–$Cmma(2c)$ | 2..–2.. | 8-gon | $4e$ $.2/m.$ $\frac{1}{4}\frac{1}{4}0$ | [.2.] | $4a$ 222 $\frac{1}{4}\frac{1}{2}0$ | [2..] | $4c$ $2/m..$ $0\frac{1}{2}0$ | [$m$..] | **VAL** |
| | | | $4g$ $mm2$ $0\frac{3}{4}z$ | [.$m$.] | $4e$ $.2/m.$ $\frac{1}{4}\frac{3}{4}$ | [.2.] | $4a$ 222 $\frac{1}{4}\frac{1}{2}1$ | [2..] | |
| | | | $4c$ $2/m..$ $0\frac{1}{2}1$ | [$m$..] | $4g$ $mm2$ $0\frac{1}{4}1\ -z$ | [.$m$.] | | | |
| $Ia\bar{3}$–$Pa\bar{3}$ | $.\bar{3}.$–$.3.$ | 18-gon | $24d$ 2.. $00\frac{1}{4}$ | [2..] | $24d$ 2.. $\frac{1}{2}0\frac{1}{4}$ | [1] | $24d$ 2.. $\frac{3}{4}0\frac{1}{2}$ | [1] | $^{\pm}$**Y** |
| | | | $24d$ 2.. $\frac{1}{2}\frac{1}{4}\frac{1}{2}$ | [2..] | $24d$ 2.. $\frac{1}{2}\frac{1}{4}0$ | [1] | $24d$ 2.. $\frac{1}{2}0\ -\frac{1}{4}$ | [1] | |
| | | | $24d$ 2.. $\frac{1}{4}00$ | [2..] | $24d$ 2.. $\frac{1}{4}\frac{1}{2}0$ | [1] | $24d$ 2.. $\frac{1}{2}\frac{3}{4}0$ | [1] | |
| | | | $24d$ 2.. $\frac{1}{2}\frac{1}{2}\frac{1}{4}$ | [2..] | $24d$ 2.. $0\frac{1}{2}\frac{1}{4}$ | [1] | $24d$ 2.. $-\frac{1}{4}\frac{1}{2}0$ | [1] | |
| | | | $24d$ 2.. $0\frac{1}{4}0$ | [2..] | $24d$ 2.. $0\frac{1}{4}\frac{1}{2}$ | [1] | $24d$ 2.. $0\frac{1}{2}\frac{3}{4}$ | [1] | |
| | | | $24d$ 2.. $\frac{1}{4}\frac{1}{2}\frac{1}{2}$ | [2..] | $24d$ 2.. $\frac{1}{4}0\frac{1}{2}$ | [1] | $24d$ 2.. $0\ -\frac{1}{4}\frac{1}{2}$ | [1] | |
| $Ia\bar{3}$–$Pa\bar{3}$ | $.\bar{3}.$–$.3.$ | 18-gon | $24d$ 2.. $00\frac{1}{4}$ | [2..] | $24d$ 2.. $\frac{1}{2}0\frac{1}{4}$ | [1] | $16c$ .3. $\frac{1}{2}\ +xx\frac{1}{2}\ -x$ | [1] | **C**($^{\pm}$**Y**) |
| | | | $24d$ 2.. $\frac{1}{2}\frac{1}{4}\frac{1}{2}$ | [2..] | $24d$ 2.. $\frac{1}{2}\frac{1}{4}0$ | [1] | $16c$ .3. $\frac{1}{2}\ -xx\ -x$ | [1] | |
| | | | $24d$ 2.. $\frac{1}{4}00$ | [2..] | $24d$ 2.. $\frac{1}{4}\frac{1}{2}0$ | [1] | $16c$ .3. $\frac{1}{2}\ -x\frac{1}{2}\ +xx$ | [1] | |
| | | | $24d$ 2.. $\frac{1}{2}\frac{1}{2}\frac{1}{4}$ | [2..] | $24d$ 2.. $0\frac{1}{2}\frac{1}{4}$ | [1] | $16c$ .3. $-x\frac{1}{2}\ -xx$ | [1] | |
| | | | $24d$ 2.. $0\frac{1}{4}0$ | [2..] | $24d$ 2.. $0\frac{1}{4}\frac{1}{2}$ | [1] | $16c$ .3. $x\frac{1}{2}\ -x\frac{1}{2}\ +x$ | [1] | |
| | | | $24d$ 2.. $\frac{1}{4}\frac{1}{2}\frac{1}{2}$ | [2..] | $24d$ 2.. $\frac{1}{4}0\frac{1}{2}$ | [1] | $16c$ .3. $x\ -x\frac{1}{2}\ -x$ | [1] | |

Table 8. *Continued*

| space-group pair $G$-$S$ | other disc-like surface patch: site sym. | polygon | vertices | | | minimal surface |
|---|---|---|---|---|---|---|
| $Ibca$–$Pbca$ | $\bar{1}$–1 | 18-gon | $8c$ 2.. $00\frac{1}{4}$ [2..] | $8c$ 2.. $\frac{1}{2}0\frac{1}{4}$ [1] | $8d$ .2. $\frac{3}{4}0\frac{1}{2}$ [1] | $\mathbf{o}^{\pm}\mathbf{Y}$ |
| | | | $8e$ ..2 $\frac{1}{2}\frac{1}{4}\frac{1}{2}$ [..2] | $8e$ ..2 $\frac{1}{2}\frac{1}{4}0$ [1] | $8c$ 2.. $\frac{1}{2}0\,-\frac{1}{4}$ [1] | |
| | | | $8d$ .2. $\frac{1}{4}00$ [.2.] | $8d$ .2. $\frac{1}{4}\frac{1}{2}0$ [1] | $8e$ ..2 $\frac{1}{2}\frac{3}{4}0$ [1] | |
| | | | $8c$ 2.. $\frac{1}{2}\frac{1}{2}\frac{1}{4}$ [2..] | $8c$ 2.. $0\frac{1}{2}\frac{1}{4}$ [1] | $8d$ .2. $-\frac{1}{4}\frac{1}{2}0$ [1] | |
| | | | $8e$ ..2 $0\frac{1}{4}0$ [..2] | $8e$ ..2 $0\frac{1}{4}\frac{1}{2}$ [1] | $8c$ 2.. $0\frac{1}{2}\frac{3}{4}$ [1] | |
| | | | $8d$ .2. $\frac{1}{4}\frac{1}{2}\frac{1}{2}$ [.2.] | $8d$ .2. $\frac{1}{4}0\frac{1}{2}$ [1] | $8e$ ..2 $0\,-\frac{1}{4}\frac{1}{2}$ [1] | |
| $Ibca$–$Pbca$ | $\bar{1}$–1 | 18-gon | $8c$ 2.. $00\frac{1}{4}$ [2..] | $8c$ 2.. $\frac{1}{2}0\frac{1}{4}$ [1] | $16f$ 1 $\frac{1}{2}+xy\frac{1}{2}-z$ [1] | $\mathbf{oC}(^{\pm}\mathbf{Y})$ |
| | | | $8e$ ..2 $\frac{1}{2}\frac{1}{4}\frac{1}{2}$ [..2] | $8e$ ..2 $\frac{1}{2}\frac{1}{4}0$ [1] | $16f$ 1 $\frac{1}{2}-xy\,-z$ [1] | |
| | | | $8d$ .2. $\frac{1}{4}00$ [.2.] | $8d$ .2. $\frac{1}{4}\frac{1}{2}0$ [1] | $16f$ 1 $\frac{1}{2}-x\frac{1}{2}+yz$ [1] | |
| | | | $8c$ 2.. $\frac{1}{2}\frac{1}{2}\frac{1}{4}$ [2..] | $8c$ 2.. $0\frac{1}{2}\frac{1}{4}$ [1] | $16f$ 1 $-x\frac{1}{2}-yz$ [1] | |
| | | | $8e$ ..2 $0\frac{1}{4}0$ [..2] | $8e$ ..2 $0\frac{1}{4}\frac{1}{2}$ [1] | $16f$ 1 $x\frac{1}{2}-y\frac{1}{2}+z$ [1] | |
| | | | $8d$ .2. $\frac{1}{4}\frac{1}{2}\frac{1}{2}$ [.2.] | $8d$ .2. $\frac{1}{4}0\frac{1}{2}$ [1] | $16f$ 1 $x\,-y\frac{1}{2}-z$ [1] | |

labyrinths. The two *labyrinth graphs* associated with an intersection-free 3-periodic surface may be constructed as follows: each labyrinth graph is entirely located within its labyrinth; each branch of a labyrinth contains an edge of its graph; each circuit of one labyrinth graph encircles at least one edge of the other. Any of the two labyrinth graphs then may be used to represent the surface, e.g. for the calculation of the genus. To this end, either the number of circuits per unit cell (of $S$ in the case of a balance surface, of $G$ otherwise), or of the corresponding finite graph embedded in $T^3$, has to be determined. Following Hyde (1989), one may separate a connected subgraph without translationally equivalent vertices from any of the labyrinth graphs by removing $p$ edges. Then

$$g = \tfrac{1}{2}p + q \tag{2.11}$$

holds, where $q$ is the number of edges that have to be removed in addition to make the subgraph simply connected. As $p$ equals at least 6, the genus of a 3-periodic surface without self-intersections must be at least 3. Fischer & Koch (1989c) have derived another formula using crystallographic terms:

$$g = 1 + \sum_i m_i(\tfrac{1}{2}e_i - 1), \tag{2.12}$$

where $m_i$ is the multiplicity of the $i$th vertex with respect to the primitive unit cell of $S$, $e_i$ is the number of edges meeting at this vertex and $i$ runs over all symmetrically inequivalent vertices of the labyrinth graph. This definition of the genus $g$ of a 3-periodic surface by means of the embedding in $T^3$, as used in most papers on the subject, has been criticized by Anderson *et al.* (1990). Their definition of the genus $g'$ of a 3-periodic surface is the number of edges per unit cell which have to be removed from the entire infinite labyrinth graph in order to be left with a simply connected infinite graph. Then each simply connected subgraph within a unit cell remains connected to two such subgraphs in adjacent cells (by analogy with the construction used to prove the countability of lattice points), resulting in

$$g' = g - 1. \tag{2.13}$$

Though this approach seems to be more natural, it has one drawback. The relation between the genus and the Euler characteristic then differs between non-periodic and periodic surfaces. $g' = 1 - \frac{1}{2}\chi$ holds only for non-periodic surfaces, whereas for periodic surfaces without boundaries it is modified to

$$g' = -\tfrac{1}{2}\chi. \tag{2.14}$$

### (*k*) *Flat points*

Each 3-periodic minimal surface contains exceptional points, called flat points, for which both main curvatures equal zero (cf. § 1). Such a point has a surrounding with $j \geqslant 3$ valleys separated by $j$ ridges, in contrast to an ordinary point on a minimal surface with a saddle-shaped surrounding with $j = 2$.

Flat points on minimal surfaces are of special interest because the normal vector at a flat point corresponds to a singular point of the Weierstrass function related to the minimal surface. They may therefore be used for the parametrization of 3-periodic minimal surfaces and for the derivation of families of metrically distorted surfaces from higher symmetry surfaces containing straight lines (Lidin & Hyde 1987; Fogden & Hyde 1992*a*, *b*; Cvijović & Klinowski 1992*a*–*c*, 1993; Fogden 1993, 1994).

The number $j$ might be used as a measure of the degree of flatness for any point

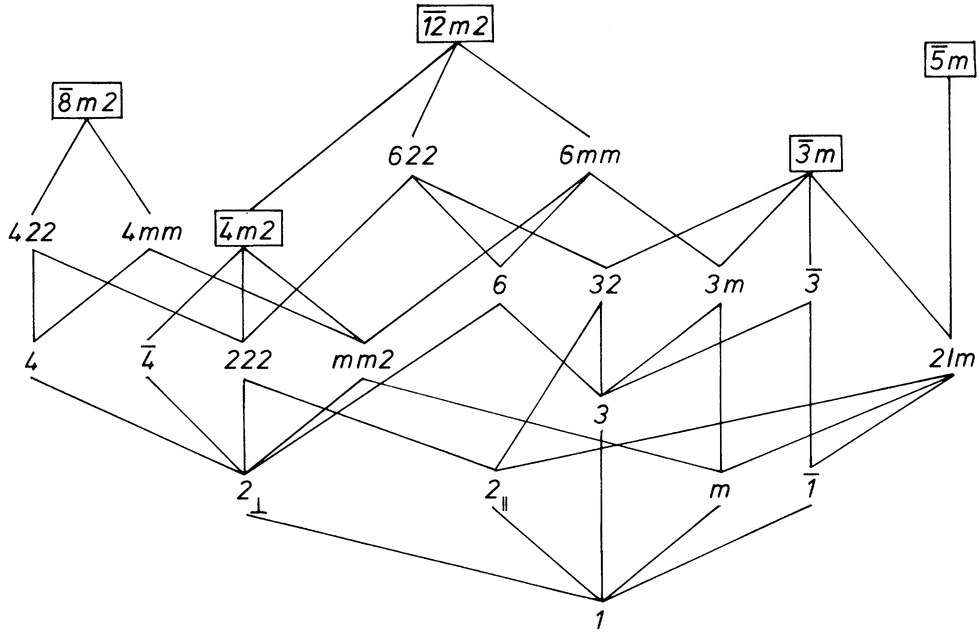


Figure 8. Group–subgroup diagram displaying the maximal site symmetry (in frame) and all possible crystallographic site symmetries for (flat) points of order $\beta \leqslant 4$.

on a minimal surface, but instead of that the *order* $\beta = j - 2$ of a point has been defined for this purpose (cf. Hyde 1989). Let $P_0$ be a point of a minimal surface and $\boldsymbol{n}_0$ the normal vector at $P_0$. Then a second point $P$ is moved on the surface around $P_0$, and the behaviour of its normal vector $\boldsymbol{n}$ during this motion is considered. If $P_0$ is an ordinary point, $\boldsymbol{n}$ rotates once around $\boldsymbol{n}_0$ during one revolution of $P_0$ around $P$. If $P_0$ is a flat point, however, $\boldsymbol{n}$ rotates more than once, say $p$ times, around $\boldsymbol{n}_0$ per revolution of $P$. Then the order $\beta$ of $P_0$ is given by

$$\beta = p - 1. \tag{2.15}$$

From the pattern of valleys and ridges follows the maximal site symmetry compatible with a point of given order. This is $\bar{4}m2$ for $\beta = 0$ (ordinary point), $\bar{3}m$ for $\beta = 1$ (monkey saddle), or, in general, $\tilde{N}m2$ or $\tilde{N}m$ for $\beta$ even or odd, respectively, with $N = 2j = 2\beta + 4$ and $\tilde{N}$ designating an $N$-fold roto*reflection*. Possible site symmetries for the different kinds of (flat) points, therefore, must be subgroups of these maximal ones. This means especially that only points with site symmetry $\bar{4}m2$, $\bar{4}$, 222, $mm2$, 2, $m$ or 1 can be ordinary points on a minimal surface. Any other site symmetry enforces the corresponding point to be a flat point of some minimal order that may be taken from the subgroup diagram in figure 8. This diagram covers only the range of $\beta \leqslant 4$, but 4 is the maximal value found so far for flat points on 3-periodic minimal surfaces.

These symmetry conditions, together with some general considerations, have been used to derive all flat points of the known minimal balance surfaces (cf. Koch & Fischer 1990).

The determination of flat points can be checked for completeness by means of an equation derived by Hyde (1989), following Hopf (1983):

$$g = 1 + \tfrac{1}{4}\sum_i \beta_i. \qquad (2.16)$$

The summation runs over all flat points within a primitive unit cell of $S$. As the genus of a 3-periodic minimal surface is at least 3, any such surface necessarily contains flat points (at least eight with $\beta = 1$ per primitive unit cell of $S$). It may be more practicable to refer to a primitive unit cell of $G$ and to carry out a summation over symmetrically inequivalent flat points only. For this, two cases have to be distinguished. If $S$ is a translation-equivalent subgroup of $G$ then

$$g = 1 + \tfrac{1}{4}\sum_i m_i\beta_i \qquad (2.17)$$

holds. If $S$ is a class-equivalent subgroup of $G$ then $g$ is given by

$$g = 1 + \tfrac{1}{2}\sum_i m_i\beta_i. \qquad (2.18)$$

### (*l*) *Results*

Information on spanning minimal surfaces generated from the surface patches described in §§ 2 *b*–*i* is collected in table 9. The families of minimal balance surfaces are arranged according to the growing complexity of the surfaces, i.e. according to increasing absolute value of the Euler characteristic $\chi$, higher symmetry being preferred for equal values of $\chi$. The reference to tables 1–8 and to cited earlier publications is given by the symbol of the family in column 1. The generating surface patches are characterized in column 2. The maximal symmetry compatible with all surfaces of the family, the group–subgroup pair $G$–$S$, is shown in column 3. The topology of each surface is described in the fourth column by its Euler characteristic $\chi$. In other columns some details on the flat points of each minimal surface are listed: the order $\beta$ and the Wyckoff position and site symmetry with respect to $G$. The last column contains the references to original papers.

Table 9 is complete in the following sense: there do not exist further spanning minimal surfaces which may be generated from skew polygons which are disc-like spanned, from catenoid-like surface patches, from branched catenoids, from multiple catenoids, or from infinite strips. It may, however, be possible to find additional 3-periodic minimal surfaces without self-intersections which contain straight lines and are generated from surface patches, other than those described in §§ 2 *b*–*i*.

## 3. Spanning minimal surfaces with self-intersections

### (*a*) *Fundamentals*

So far, spanning minimal surfaces with self-intersections have only been derived from disc-like surface patches spanning skew polygons. Examples are given by Stessmann (1934), Karcher (1989) and Schoen (1970), but only Schoen described the geometrical properties of the one such surface he had derived. He states that his surface is non-orientable and subdivides $R^3$ into two congruent labyrinths, similar to those of an intersection-free minimal surface. It is an example of the kind of surfaces which will be discussed below: spanning minimal surfaces with self-intersections entirely along straight lines. Recently, procedures for a systematic treatment of such

Table 9. *Spanning minimal surfaces without self-intersections*

| minimal surface | surface patch | space-group pair $G$–$S$ | $\chi$ | flat points $\beta$ | flat points Wyck. position | flat points symmetry | references |
|---|---|---|---|---|---|---|---|
| **P** | 4-gon | $Im\bar{3}m$–$Pm\bar{3}m$ | −4 | 1 | $8c$ | $.\bar{3}m$ | Schwarz 1890; Fischer & Koch 1987 |
| **D** | 4-gon | $Pn\bar{3}m$–$Fd\bar{3}m(2a)$ | −4 | 1 | $4c$ | $.\bar{3}m$ | Schwarz 1890; Fischer & Koch 1987 |
| **H** | 3-catenoid | $P6_3/mmc$–$P\bar{6}m2$ | −4 | 1 | $2a$ | $\bar{3}m.$ | Schwarz 1890; |
| | | | | 1 | $6g$ | $.2/m.$ | Koch & Fischer 1988 |
| **rPD** | 3-catenoid | $R\bar{3}m$–$R\bar{3}m(2c)$ | −4 | 1 | $3b$ | $\bar{3}m$ | Schwarz 1890; |
| | | | | 1 | $9d$ | $.2/m$ | Koch & Fischer 1988 |
| **CLP** | 6-gon | $P4_2/mcm$–$P4_2/mmc(v)$ | −4 | 1 | $4f$ | $2/m..$ | Schwarz 1890; Koch & Fischer 1988 |
| **tD** | 5-gon | $P4_2/nnm$–$I4_1/amd(v, 2c)$ | −4 | 1 | $4f$ | $..2/m$ | Schwarz 1890; Koch & Fischer 1988 |
| **tP** | 4-catenoid | $I4/mmm$–$P4/mmm$ | −4 | 1 | $8f$ | $..2/m$ | Schoen 1970; Koch & Fischer 1988 |
| **oCLP** | 6-gon | $Pccm$–$Cccm(2a, 2b)$ | −4 | 1 | $2c$ | $..2/m$ | Koch & Fischer 1988 |
| | | | | 1 | $2d$ | $..2/m$ | |
| **oDa** | 6-gon | $Pnnn$–$Fddd(2a, 2b, 2c)$ | −4 | 1 | $4f$ | $\bar{1}$ | Schoen 1970; Koch & Fischer 1988 |
| **oDb** | 8-gon | $Cmma$–$Imma(2c)$ | −4 | 1 | $4d$ | $2/m..$ | Koch & Fischer 1988 |
| | | | | 1 | $4e$ | $.2/m.$ | |
| **oPb** | 4-catenoid | $Fmmm$–$Cmmm$ | −4 | 1 | $8c$ | $2/m..$ | Schoen 1970; |
| | | | | 1 | $8d$ | $.2/m.$ | Koch & Fischer 1988 |

Table 9. Cont. *Spanning minimal surfaces without self-intersections*

| minimal surface | surface patch | space-group pair $G$–$S$ | $\chi$ | flat points $\beta$ | Wyck. position | symmetry | references |
|---|---|---|---|---|---|---|---|
| **HS2** | 8-gon | $P6_222$–$P3_212$ | −6 | 1 | $6f$ | 2.. | Koch & Fischer 1988 |
| | | | | 1 | $6h$ | .2. | |
| **MC5** | double catenoid | $P4_2/mcm$–$Cmmm(v)$ | −8 | 1 | $8m$ | .2. | Karcher 1989; |
| | | | | 1 | $8n$ | $m$.. | Koch & Fischer 1989*a* |
| **oMC5** | double catenoid | $Pccm$–$P2/m$ | −8 | 1 | $4j$ | 2.. | Koch & Fischer 1989*a* |
| | | | | 1 | $4l$ | .2. | |
| | | | | 1 | $4q$ (2×) | ..$m$ | |
| **PT** | 4-catenoid with 2 spouts | $Fmmm$–$Cmmm$ | −8 | 1 | $8c$ | $2/m$.. | Koch & Fischer 1989*b* |
| | | | | 1 | $8d$ | .$2/m$. | |
| | | | | 1 | $16o$ | ..$m$ | |
| **VAL** | other surface patch | $Cmma$–$Cmma(2c)$ | −8 | 3 | $4e$ | $2/m$.. | Fogden & Hyde 1992*b* |
| | | | | 1 | $4c$ | .$2/m$. | |
| **BC3** | 4-fold branched catenoid | $I422$–$I4$ | −10 | 2 | $2a$ | 422 | Fischer & Koch 1989*a* |
| | | | | 2 | $2b$ | 422 | |
| | | | | 1 | $8i$ | .2. | |
| | | | | 1 | $8j$ | ..2 | |
| | | | | 1 | $16k$ | 1 | |
| **HS1** | 8-gon | $P6_222$–$P6_122(2c)$ | −12 | 1 | $6f$ | 2.. | Koch & Fischer 1988 |
| | | | | 1 | $6h$ | .2. | |
| **HS3** | 4-catenoid | $P6_222$–$P6_422(2c)$ | −12 | 1 | $6h$ | .2. | Koch & Fischer 1988 |
| | | | | 1 | $6j$ | ..2 | |

Table 9. Cont. *Spanning minimal surfaces without self-intersections*

| minimal surface | surface patch | space-group pair $G$–$S$ | $\chi$ | flat points $\beta$ | flat points Wyck. position | flat points symmetry | references |
|---|---|---|---|---|---|---|---|
| **MC1** | triple catenoid | $P6_3/mcm$–$P\bar{6}2m$ | −12 | 1 | $2b$ | $\bar{3}.m$ | Karcher 1989; Koch & Fischer 1989*a* |
| | | | | 1 | $4d$ | 3.2 | |
| | | | | 1 | $6f$ | $..2/m$ | |
| | | | | 1 | $12j$ | $m..$ | |
| **ST1** | zigzag strip | $P6_222$–$P6_422(2c)$ | −12 | 1 | $12k$ | 1 | Fischer & Koch 1989*b* |
| **C(H)** | 3-catenoid with 3 spouts | $P6_3/mmc$–$P\bar{6}m2$ | −12 | 1 | $2a$ | $\bar{3}m.$ | Schoen 1970; Koch & Fischer 1989*b* |
| | | | | 1 | $4f$ | $3m.$ | |
| | | | | 1 | $6g$ | $.2/m.$ | |
| | | | | 1 | $12j$ | $m..$ | |
| **BC2** | 2-fold branched catenoid | $P4_2/nnm$–$P4_2nm$ | −12 | 1 | $4e$ | $..2/m$ | Fischer & Koch 1989*a* |
| | | | | 1 | $4f$ | $..2/m$ | |
| | | | | 1 | $8j$ | $.2.$ | |
| | | | | 1 | $8m$ | $..m$ | |
| **ST2** | zigzag/ meander strip | $P4_2/nbc$–$P4_2/n$ | −12 | 1 | $8h$ | $.2.$ | Fischer & Koch 1989*b* |
| | | | | 1 | $16k$ | 1 | |
| **C(P)** | 8-gon | $Im\bar{3}m$–$Pm\bar{3}m$ | −16 | 2 | $12e$ | $4m.m$ | Neovius 1883; Fischer & Koch 1987 |
| | | | | 1 | $8c$ | $.\bar{3}m$ | |
| **BC1** | 3-fold branched catenoid | $P6_322$–$P6_3$ | −16 | 1 | $2a$ | $32.$ | Fischer & Koch 1989*a* |
| | | | | 1 | $2b$ | 3.2 | |
| | | | | 1 | $2c$ | 3.2 | |
| | | | | 1 | $2d$ | 3.2 | |
| | | | | 1 | $6g$ | $.2.$ | |
| | | | | 1 | $6h$ | $..2$ | |
| | | | | 1 | 12i | 1 | |

Table 9. Cont. *Spanning minimal surfaces without self-intersections*

| minimal surface | surface patch | space-group pair $G$–$S$ | $\chi$ | flat points $\beta$ | Wyck. position | symmetry | references |
|---|---|---|---|---|---|---|---|
| **R2** | 3-catenoid | $I4/mcm$–$P4/mbm$ | −16 | 2 | $4a$ | 422 | Schoen 1970; |
| | | | | 1 | $8e$ | $..2/m$ | Koch & Fischer 1988 |
| | | | | 1 | $16j$ | $.2.$ | |
| **MC6** | double catenoid | $I4/mcm$–$P4/mbm$ | −16 | 2 | $4a$ | 422 | Karcher 1989; |
| | | | | 1 | $8e$ | $..2/m$ | Koch & Fischer 1989*a* |
| | | | | 1 | $16k$ | $m..$ | |
| **MC7** | quadruple catenoid | $P4/mcc$–$P4/m$ | −16 | 2 | $2a$ | 422 | Karcher 1989; |
| | | | | 2 | $2c$ | 422 | Koch & Fischer 1989*a* |
| | | | | 1 | $8l$ | $.2.$ | |
| | | | | 1 | $8m(2\times)$ | $m..$ | |
| **tC(P)** | 4-catenoid with 4 spouts | $I4/mmm$–$P4/mmm$ | −16 | 2 | $4e$ | $4mm$ | Koch & Fischer 1989*b* |
| | | | | 1 | $8f$ | $..2/m$ | |
| | | | | 1 | $16l$ | $m..$ | |
| **oC(P)** | 4-catenoid with 4 spouts | $Fmmm$–$Cmmm$ | −16 | 2 | $8i$ | $mm2$ | Koch & Fischer 1989*b* |
| | | | | 1 | $8c$ | $2/m..$ | |
| | | | | 1 | $8d$ | $.2/m.$ | |
| | | | | 1 | $16o\ (2\times)$ | $..m$ | |
| **S** | 12-gon | $Ia\bar{3}d$–$I\bar{4}3d$ | −20 | 1 | $16a$ | $.\bar{3}.$ | Fischer & Koch 1987 |
| | | | | 1 | $16b$ | $.32$ | |
| | | | | 1 | $48g$ | $..2$ | |
| **C(Y)** | 9-gon | $I4_132$–$P4_332$ | −24 | 1 | $8b$ | $.32$ | Fischer & Koch 1987 |
| | | | | 1 | $16e$ | $.3.$ | |
| | | | | 1 | $24f$ | $2..$ | |

Table 9. Cont. *Spanning minimal surfaces without self-intersections*

| minimal surface | surface patch | space-group pair $G$–$S$ | $\chi$ | flat points $\beta$ | Wyck. position | symmetry | references |
|---|---|---|---|---|---|---|---|
| **C($^{\pm}$Y)** | other surface patch | $Ia\bar{3}$–$Pa\bar{3}$ | −24 | 1 | $8b$ | $.\bar{3}.$ | Koch & Fischer 1988; Fischer & Koch 1989*c* |
| | | | | 1 | $16c$ | .3. | |
| | | | | 1 | $24d$ | 2.. | |
| **oC(F)** | other surface patch | $Ibca$–$Pbca$ | −24 | 1 | $8b$ | $\bar{1}$ | Fischer & Koch 1990 |
| | | | | 1 | $8c$ | 2.. | |
| | | | | 1 | $8d$ | .2. | |
| | | | | 1 | $8e$ | ..2 | |
| | | | | 1 | $16f$ | 1 | |
| **R3** | 3-catenoid | $P6/mcc$–$P6/m$ | −24 | 4 | $2a$ | 622 | Schoen 1970; Koch & Fischer 1988 |
| | | | | 1 | $4c$ | 3.2 | |
| | | | | 1 | $12j$ | .2. | |
| | | | | 1 | $12k$ (2×) | ..2 | |
| **MC2** | double catenoid | $P6/mcc$–$P6/m$ | −24 | 4 | $2a$ | 622 | Karcher 1989; Koch & Fischer 1989*a* |
| | | | | 1 | $4c$ | 3.2 | |
| | | | | 1 | $12k$ | ..2 | |
| | | | | 1 | $12l$ (2×) | $m$.. | |
| **MC3** | triple catenoid | $P6/mcc$–$P6/m$ | −24 | 4 | $2a$ | 622 | Karcher 1989; Koch & Fischer 1989*a* |
| | | | | 1 | $4c$ | 3.2 | |
| | | | | 1 | $12j$ | .2. | |
| | | | | 1 | $12l$ (2×) | $m$.. | |
| **MC4** | sextuple catenoid | $P6/mcc$–$P6/m$ | −24 | 4 | $2a$ | 622 | Karcher 1989; Koch & Fischer 1989*a* |
| | | | | 1 | $4c$ | 3.2 | |
| | | | | 1 | $12k$ | ..2 | |
| | | | | 1 | $12l$ (2×) | $m$.. | |

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Table 9. Cont. *Spanning minimal surfaces without self-intersections*

| minimal surface | surface patch | space-group pair $G$–$S$ | $\chi$ | flat points $\beta$ | flat points Wyck. position | flat points symmetry | references |
|---|---|---|---|---|---|---|---|
| **C(P)/H** | 8-gon with spout | $Im\bar{3}m$–$Pm\bar{3}m$ | −28 | 1 | $8c$ | $.\bar{3}m$ | Karcher & Polthier 1990; |
| | | | | 1 | $48k$ | $..m$ | Koch & Fischer 1993*b* |
| **C(D)** | 12-gon | $Pn\bar{3}m$–$Fd\bar{3}m(2a)$ | −36 | 4 | $8e$ | $.3m$ | Schoen 1970; |
| | | | | 1 | $4c$ | $.\bar{3}m$ | Fischer & Koch 1987 |
| $^{\pm}$**Y** | other surface patch | $Ia\bar{3}$–$Pa\bar{3}$ | −40 | 2 | $24d$ | $2..$ | Fischer & Koch 1987, 1989*c* |
| | | | | 1 | $8b$ | $.\bar{3}.$ | |
| | | | | 1 | $24d$ | $2..$ | |
| **o**$^{\pm}$**Y** | other surface patch | $Ibca$–$Pbca$ | −40 | 2 | $8c$ | $2..$ | Koch & Fischer 1988; |
| | | | | 2 | $8d$ | $.2.$ | Fischer & Koch 1990 |
| | | | | 2 | $8e$ | $..2$ | |
| | | | | 1 | $8b$ | $\bar{1}$ | |
| | | | | 1 | $8c$ | $2..$ | |
| | | | | 1 | $8d$ | $.2.$ | |
| | | | | 1 | $8e$ | $..2$ | |
| **C(Y)/H** | 9-gon with spout | $I4_132$–$P4_332$ | −40 | 1 | $8b$ | $.32$ | Koch & Fischer 1993*b* |
| | | | | 1 | $24f$ | $2..$ | |
| | | | | 1 | $48i$ | $1$ | |

Table 9. Cont. *Spanning minimal surfaces without self-intersections*

| minimal surface | surface patch | space-group pair $G$–$S$ | $\chi$ | flat points: $\beta$ | flat points: Wyck. position | flat points: symmetry | references |
|---|---|---|---|---|---|---|---|
| **C(R2)** | 3-catenoid with 3 spouts | $I4/mcm$–$P4/mbm$ | −48 | 2 | $4a$ | 422 | Koch & Fischer 1989*b* |
| | | | | 1 | $8e$ | $..2/m$ | |
| | | | | 1 | $16j$ | $.2.$ | |
| | | | | 1 | $16k$ (3×) | $m..$ | |
| | | | | 1 | $16l$ | $..m$ | |
| **C(D)/H** | 12-gon with spout | $Pn\bar{3}m$–$Fd\bar{3}m(2a)$ | −52 | 1 | $4c$ | $.\bar{3}m$ | Koch & Fischer 1993*b* |
| | | | | 1 | $24k$ (2×) | $..m$ | |
| **C(R3)** | 3-catenoid with 3 spouts | $P6/mcc$–$P6/m$ | −72 | 4 | $2a$ | 622 | Koch & Fischer 1989*b* |
| | | | | 1 | $4c$ | 3.2 | |
| | | | | 1 | $12j$ | $.2.$ | |
| | | | | 1 | $12k$ (2×) | $..2$ | |
| | | | | 1 | $12l$ (6×) | $m..$ | |
| | | | | 1 | $24m$ | 1 | |

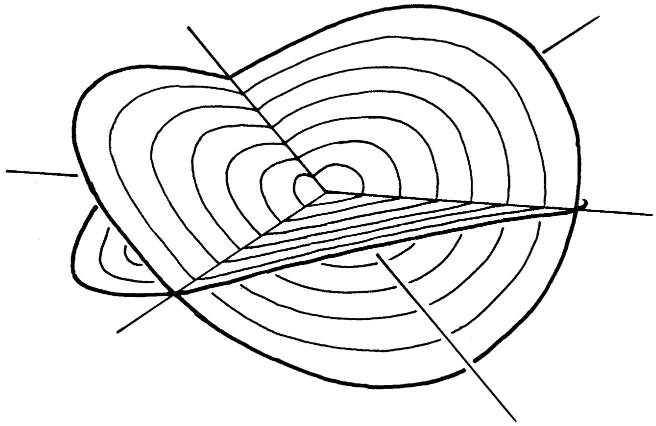

Figure 9. Neighbourhood of a branch point with site symmetry 32.

surfaces have been proposed (Koch 1995) and several new examples described (Fischer & Koch 1995, 1996).

Each skew polygon, all edges of which correspond to rotation axes (except 3-fold ones) of some space group $G$, generates a 3-periodic minimal surface. Normally this surface shows self-intersections, if the polygon violates the rules described in §§ 2 *a*, *b*. In the exceptional case, however, where the polygon in question can be subdivided by further 2-fold axes into smaller ones listed already in table 1, the generated surface is a minimal balance surface (cf. Fischer & Koch 1996).

If an edge of a skew polygon corresponds to a 4-fold or 6-fold axis of $G$, the generated infinite surface must intersect itself along this axis. In the case of a 4-fold axis the surface is necessarily non-orientable, in the case of a 6-fold axis it may be orientable or non-orientable. Examples have been found for all three situations.

A special situation arises if the skew polygon has a vertex angle of 120°. The continuation of such a surface patch results in a piece of an infinite minimal surface winding twice around that vertex (cf. figure 9). The surface then intersects itself along each of the 2-fold axes through the 120° vertex, but only on one side of it, whereas on its other side these axes are not contained in the surface. Such a vertex has at least site symmetry 32 and is called a *branch point* of the surface (cf. Schoen 1970; Nitsche 1989; Dierkes *et al.* 1992). It does not fulfill the general condition for the mean curvature of a minimal surface. Branch points may occur as well in orientable as in non-orientable minimal surfaces.

Visualizing a 3-periodic surface is even more difficult if it intersects itself than if it is free of self-intersections. Often a full insight into its geometrical properties is only achieved from a model. But before taking the trouble of building such a model it is worthwhile to derive as many properties of the infinite surface as possible from the information already contained in the surface patch, i.e. in the skew polygon spanned by the patch. To this end, abstract procedures have been developed to determine the following properties (Koch 1995; Fischer & Koch 1996).

(i) *Full symmetry group G of the surface*

The 2-fold rotations corresponding to the edges of the generating skew polygon together with the point-group symmetry of the surface patch generate a group, the space group $G$ describing the full symmetry of the infinite surface.





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(ii) *Edges with self-intersection*

In order to decide whether or not self-intersection occurs along a certain polygon edge, the number $n$ of surface patches sharing that edge is calculated. For this, the edge has to be related to its Wyckoff position in $G$. Referred to one unit cell of $G$, the length of this edge (eventually together with other ones from the same Wyckoff position) corresponds to a certain part of this Wyckoff position. This fraction multiplied by the number of surface patches per unit cell of $G$ then gives $n$. If $n = 2$, there is no self-intersection. If $n = 4$, 6 or 12, two, three or six pieces of the surface, respectively, intersect each other in the corresponding edge.

Only if all edges of the generating skew polygon correspond to lines with self-intersection, the infinite surface may subdivide $R^3$ into spatial subunits of two different kinds. Otherwise, all subunits are necessarily symmetrically equivalent and, therefore, congruent.

(iii) *Orientability*

By analogy to minimal balance surfaces, the symmetry of an orientable spanning minimal surface with self-intersections may also be described by a group–subgroup pair of space groups $G$–$S$ with index 2. Again $G$ consists of all symmetry operations mapping the surface onto itself, whereas $S$ contains only those operations which do not interchange the two sides of the surface. In particular, $S$ must not contain any of the 2-fold rotations referring to edges of the generating skew polygon.

In the case of a non-orientable surface there does not exist any subgroup $S$ of the full symmetry group $G$ of the surface with the above properties. This difference may be used to decide whether or not a surface is orientable. For each subgroup of $G$ with index 2 it has to be checked which of the 2-fold rotations corresponding to polygon edges are still contained in the subgroup. If there exists a subgroup $S$ where all these 2-fold rotations are missing, then the surface is an orientable spanning surface with symmetry $G$–$S$. If no such subgroup exists the surface is non-orientable.

In the second independent procedure to decide the orientability of a surface, closed rings of surface patches are examined. If the surface is non-orientable, then there must exist rings which are twisted like a Möbius strip. As each surface patch of a ring is related to its two neighbours by 2-fold rotations, closing up to a normal ring is only achieved for an even number of patches, whereas each ring formed by an odd number of surface patches is necessarily twisted. The search for odd-membered rings may be done by looking for odd products of those 2-fold rotations that map neighbouring surface patches onto each other. If there exists such an odd product equal to the identity, then the infinite surface is non-orientable. The number $l$ of surface patches in the shortest twisted ring may be used to characterize a non-orientable spanned minimal surface.

(iv) *Symmetry and periodicity of the spatial subunits*

Each 3-periodic minimal surface subdivides $R^3$ into spatial subunits. If the surface intersects itself, one would expect, on first thought, that these subunits should be polyhedra with curved faces. Schoen's example (1970) shows, however, that the subunits may also be 3-periodic labyrinths.

To derive the symmetry group $U$ of a spatial subunit, a set of generators for $U$ is constructed. For this, we have to distinguish between polygon edges with and without self-intersection. Rotation around any edge without self-intersection maps one subunit onto a neighbouring unit and a second rotation of the same kind maps





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the second subunit back onto the first. Accordingly, all products $g_1 g_2$ and $g_2 g_1$ of two 2-fold rotations $g_1$ and $g_2$, which correspond to polygon edges without self-intersection, belong to the symmetry group of the spatial subunit in question. With respect to polygon edges with self-intersections, two cases have to be distinguished: symmetrically equivalent sides of adjacent surface patches may be oriented either towards the same spatial subunit or towards different spatial subunits. In the first case, there always exists a symmetry operation, e.g. a mirror reflection, which maps the neighbouring surface patches onto each other and the particular spatial subunit onto itself. The second case occurs, for example, if the polygon edge corresponds to a 4- or 6-fold rotation axis. Then all products $g_1 g_2$ of such a 4- or 6-fold rotation $g_1$ with a 2-fold rotation $g_2$ corresponding to an edge without self-intersection belong to the symmetry group $U$ of the spatial subunit considered. The set of generators for $U$ is completed by those site-symmetry operations of the surface patch which do not interchange its sides.

If $U$ again is a space group, then the surface subdivides $R^3$ into $i$ congruent 3-periodic labyrinths, where $i$ is the index of $U$ in $G$. Labyrinths may occur as well in connection with orientable as with non-orientable surfaces. So far, two, four, and eight congruent mutually interpenetrating labyrinths have been found. The symmetry of each self-intersecting minimal surface with a finite number of differently coloured labyrinths may be expressed as a group–subgroup pair $G$–$U$ of space groups or instead by a multicolour space group.

If $U$ is a point group, a rod group, or a layer group, then there exist infinitely many spatial subunits, i.e. finite subunits ('polyhedra'), 1-periodic subunits ('tubes'), or 2-periodic subunits ('flat labyrinths'), respectively. Examples for all these cases have been found.

### (*b*) *Results*

The procedures described in § 3 *a* have been applied to a number of skew polygons. First, the polygon given by Schoen (1970) for his non-orientable surface was identified as formed by 2-fold axes of space group $I4_132$, and most of his statements could be confirmed. Thus his surface is indeed non-orientable and subdivides $R^3$ into two congruent labyrinths. Then those skew polygons described by Stessmann (1934) and by Karcher (1989) that do not fulfil the conditions for an intersection-free surface have been analysed likewise. Finally, several additional polygons made up by even-fold rotation axes within some space group have been studied. It has to be stressed that here completeness has certainly not been achieved, as was possible for the disc-like surface patches generating intersection-free spanning minimal surfaces (cf. § 2 *b*). The results in tables 10–12 reflect the current state of knowledge.

This situation is due to the fact that the exclusion of self-intersections, other than along straight lines, can only be done so far by geometric inspection. No abstract procedure could be developed that limits the size of a skew polygon under this restriction. On the other hand, there exist small skew polygons, as for example the 4-gon IV of Stessmann (1934), which generate surfaces with additional self-intersections.

By analogy to table 1, table 10 gives information on eight families of orientable minimal surfaces. The symmetry of each surface is described in column 1 by the group–subgroup pair $G$–$S$ (cf. § 3 *a*). The next columns refer to the corresponding disc-like surface patch. A number $\langle n \rangle$ between the description of two vertices of the generating skew polygon indicates that $n$ pieces of the surface are intersecting

Table 10. *Orientable spanning minimal surfaces with self-intersections entirely along straight lines*

| space-group pair $G$-$S$ | disc-like surface patch: site sym. | polygon | vertices | | | | | | spatial subunits: period. | sym. $U$ | minimal surface |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P6/mcc$– | $m..$ | 6-gon | $2a\ 622\ 00\frac{1}{4}$ | $\langle 3\rangle$ | $2a\ 622\ 00\frac{3}{4}$ | | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\frac{3}{4}$ | | 3 | $P\bar{6}2c$ | **HAT1–3$^2$** |
| $P\bar{6}2c$ | | | $6f\ 222\ \frac{1}{2}0\frac{3}{4}$ | | $6f\ 222\ \frac{1}{2}0\frac{1}{4}$ | | $4c\ 3.2\ \frac{2}{3}\frac{1}{3}\frac{1}{4}$ | | | | |
| $P622$– | 1 | 5-gon | $1a\ 622\ 000$ | $\langle 3\rangle$ | $1b\ 622\ 00\frac{1}{2}$ | | $3g\ 222\ \frac{1}{2}0\frac{1}{2}$ | | 3 | $P6_322(2c)$ | **HAT2–3$^2$** |
| $P6_322(2c)$ | | | $3f\ 222\ \frac{1}{2}00$ | | $2c\ 3.2\ \frac{2}{3}\frac{1}{3}0$ | | | | | | |
| $Pm\bar{3}n$–$Pm\bar{3}$ | $mm2..$ | 8-gon | $6d\ \bar{4}m.2\ 0\frac{1}{4}\frac{1}{2}$ | $\langle 2\rangle$ | $8e\ .32\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$ | $\langle 2\rangle$ | $6d\ \bar{4}m.2\ \frac{1}{4}\frac{1}{2}0$ | $\langle 2\rangle$ | 1 | $P\bar{4}(m.2)$ | **WI-10** |
| | | | $8e\ .32\ \frac{1}{4}\frac{3}{4}\frac{1}{4}$ | $\langle 2\rangle$ | $6d\ \bar{4}m.2\ 0\frac{3}{4}\frac{1}{2}$ | $\langle 2\rangle$ | $8e\ .32\ \frac{1}{4}\frac{3}{4}\frac{3}{4}$ | $\langle 2\rangle$ | 0 | $m\bar{3}.$ | |
| | | | $6d\ \bar{4}m.2\ \frac{1}{4}\frac{1}{2}1$ | $\langle 2\rangle$ | $8e\ .32\ \frac{1}{4}\frac{1}{4}\frac{3}{4}$ | $\langle 2\rangle$ | | | | | |
| $P6/mmm$– | $mm2$ | 6-gon | $1a\ 6/mmm\ 000$ | $\langle 2\rangle$ | $3f\ mmm\ \frac{1}{2}00$ | $\langle 2\rangle$ | $3f\ mmm\ \frac{1}{2}01$ | $\langle 2\rangle$ | 1 | $P\bar{6}(m2)$ | **GP-10** |
| $P\bar{6}m2$ | | | $1a\ 6/mmm\ 001$ | $\langle 2\rangle$ | $3f\ mmm\ \frac{1}{2}\frac{1}{2}1$ | $\langle 2\rangle$ | $3f\ mmm\ \frac{1}{2}\frac{1}{2}0$ | $\langle 2\rangle$ | 0 | $6/mmm$ | |
| $P6/mmm$ | $mm2$ | 6-gon | $3f\ mmm\ \frac{1}{2}00$ | $\langle 2\rangle$ | $2c\ \bar{6}m2\ \frac{2}{3}\frac{1}{3}0$ | $\langle 2\rangle$ | $3f\ mmm\ \frac{1}{2}\frac{1}{2}0$ | $\langle 2\rangle$ | 0 | $\bar{6}m2$ | **GP-01** |
| $P6_3/mmc(2c)$ | | | $3f\ mmm\ \frac{1}{2}\frac{1}{2}1$ | $\langle 2\rangle$ | $2c\ \bar{6}m2\ \frac{2}{3}\frac{1}{3}1$ | $\langle 2\rangle$ | $3f\ mmm\ \frac{1}{2}01$ | $\langle 2\rangle$ | 1 | $P6/m(mm)$ | |
| $Pm\bar{3}m$– | $..m$ | 4-gon | $1a\ m\bar{3}m\ 000$ | $\langle 2\rangle$ | $3c\ 4/mm.m\ \frac{1}{2}0\frac{1}{2}$ | $\langle 2\rangle$ | $3d\ 4/mm.m\ 00\frac{1}{2}$ | $\langle 2\rangle$ | 0 | $4m.m$ | **CAB-00** |
| $Fm\bar{3}m(2a)$ | | | $3c\ 4/mm.m\ 0\frac{1}{2}\frac{1}{2}$ | $\langle 2\rangle$ | | | | | 0 | $m\bar{3}m$ | |
| $P6/mmm$– | $.m.$ | 6-gon | $1a\ 6/mmm\ 000$ | $\langle 2\rangle$ | $3f\ mmm\ \frac{1}{2}00$ | $\langle 2\rangle$ | $3g\ mmm\ \frac{1}{2}0\frac{1}{2}$ | $\langle 2\rangle$ | 0 | $\bar{6}m2$ | **GP-00** |
| $P6_3/mmc(2c)$ | | | $2d\ \bar{6}m2\ \frac{2}{3}\frac{1}{3}\frac{1}{2}$ | $\langle 2\rangle$ | $3g\ mmm\ \frac{1}{2}\frac{1}{2}\frac{1}{2}$ | $\langle 2\rangle$ | $3f\ mmm\ \frac{1}{2}\frac{1}{2}0$ | $\langle 2\rangle$ | 0 | $6/mmm$ | |
| $P6/mmm$– | $.m.$ | 5-gon | $1a\ 6/mmm\ 000$ | $\langle 2\rangle$ | $1a\ 6/mmm\ 100$ | $\langle 2\rangle$ | $1b\ 6/mmm\ 10\frac{1}{2}$ | $\langle 2\rangle$ | 0 | $mmm$ | **NG-00** |
| $P6_3/mmc(2c)$ | | | $2d\ \bar{6}m2\ \frac{2}{3}\frac{1}{3}\frac{1}{2}$ | $\langle 2\rangle$ | $1b\ 6/mmm\ 00\frac{1}{2}$ | $\langle 6\rangle$ | | | 0 | $\bar{6}m2$ | |

Table 11. *Non-orientable spanning minimal surfaces with self-intersections entirely along straight lines subdividing $R^3$ into i labyrinths*

| surface sym. $G$ | site sym. | polygon | vertices (disc-like surface patch) | | | $l$ | $i$ | sym. $U$ (labyrinths) | minimal surface |
|---|---|---|---|---|---|---|---|---|---|
| $I432$ | ..2 | 4-gon | $2a$ 432 000<br>$6b$ 42.2 $0\frac{1}{2}0$ ⟨2⟩ | $6b$ 42.2 $\frac{1}{2}\frac{1}{2}0$ ⟨2⟩ | $2a$ 432 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ | 11 | 2 | $P4_232$ | **WF-3$^2$** |
| $I4_132$ | 1 | 5-gon | $8a$ .32 $\frac{1}{8}\frac{1}{8}\frac{1}{8}$<br>$8b$ .32 $\frac{1}{8}\frac{5}{8}-\frac{1}{8}$ ⟨2⟩ | $12c$ 2.22 $\frac{1}{4}\frac{1}{8}0$<br>$12d$ 2.22 $\frac{1}{8}\frac{1}{2}-\frac{1}{4}$ | $12d$ 2.22 $\frac{1}{4}\frac{5}{8}0$ ⟨2⟩ | 7 | 2 | $P4_332$ | **CBP1-3$^2$** |
| $P6/mcc$ | $m$.. | 8-gon | $2a$ 622 $00\frac{1}{4}$<br>$6f$ 222 $\frac{1}{2}\frac{1}{2}\frac{1}{4}$<br>$6f$ 222 $\frac{1}{2}0\frac{3}{4}$ | $6f$ 222 $\frac{1}{2}0\frac{1}{4}$ ⟨2⟩<br>$6f$ 222 $\frac{1}{2}\frac{1}{2}\frac{3}{4}$ ⟨2⟩<br>$2a$ 622 $00\frac{3}{4}$ ⟨3⟩ | $4c$ 3.2 $\frac{2}{3}\frac{1}{3}\frac{1}{4}$ ⟨2⟩<br>$4c$ 3.2 $\frac{2}{3}\frac{1}{3}\frac{3}{4}$ ⟨2⟩ | 5 | 2 | $P\bar{6}c2$ | **HBP1-3$^2$** |
| $P622$ | 1 | 6-gon | $1a$ 622 000<br>$3f$ 222 $\frac{1}{2}\frac{1}{2}0$ | $3f$ 222 $\frac{1}{2}00$ ⟨2⟩<br>$3g$ 222 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ | $2c$ 3.2 $\frac{2}{3}\frac{1}{3}0$ ⟨2⟩<br>$1b$ 622 $00\frac{1}{2}$ ⟨3⟩ | 5 | 2 | $P312$ | **HBP2-3$^2$** |
| $P622$ | ..2 | 8-gon | $1a$ 622 000<br>$1a$ 622 110 ⟨6⟩<br>$3f$ 222 $\frac{1}{2}\frac{1}{2}1$ | $2c$ 3.2 $\frac{2}{3}\frac{1}{3}0$<br>$1a$ 622 111<br>$1a$ 622 001 ⟨6⟩ | $3f$ 222 $\frac{1}{2}\frac{1}{2}0$<br>$2c$ 3.2 $\frac{2}{3}\frac{1}{3}1$ | 7 | 2 | $P622(2c)$ | **HAS1-3$^2$** |
| $P622$ | ..2 | 8-gon | $1a$ 622 000<br>$3f$ 222 $\frac{1}{2}01$<br>$3f$ 222 $\frac{1}{2}\frac{1}{2}1$ ⟨2⟩ | $2c$ 3.2 $\frac{2}{3}\frac{1}{3}0$<br>$1a$ 622 001<br>$3f$ 222 $\frac{1}{2}\frac{1}{2}0$ | $3f$ 222 $\frac{1}{2}00$ ⟨2⟩<br>$2c$ 3.2 $\frac{2}{3}\frac{1}{3}1$ | 5 | 2 | $P622(2c)$ | **ALB-3$^2$** |
| $P622$ | 1 | 7-gon | $1a$ 622 000<br>$3f$ 222 $\frac{1}{2}\frac{1}{2}0$<br>$1b$ 622 $00\frac{1}{2}$ ⟨3⟩ | $3f$ 222 $\frac{1}{2}00$ ⟨2⟩<br>$3g$ 222 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ | $2c$ 3.2 $\frac{2}{3}\frac{1}{3}0$ ⟨2⟩<br>$2d$ 3.2 $\frac{1}{3}\frac{2}{3}\frac{1}{2}$ | 5 | 2 | $P6_322(2c)$ | **HBP3-3$^2$** |

Table 11. Cont. *Non-orientable spanning minimal surfaces with self-intersections entirely along straight lines subdividing* $R^3$ *into* $i$ *labyrinths*

| surface sym. $G$ | disc-like surface patch: site sym. | polygon | vertices | | | labyrinths: $l$ | $i$ | sym. $U$ | minimal surface |
|---|---|---|---|---|---|---|---|---|---|
| $P622$ | 1 | 7-gon | $1a$ 622 000<br>$3f$ 222 $\frac{1}{2}\frac{1}{2}0$<br>$1b$ 622 $00\frac{1}{2}$ ⟨3⟩ | $3f$ 222 $\frac{1}{2}00$ ⟨2⟩<br>$3g$ 222 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ | $2c$ 3.2 $\frac{2}{3}\frac{1}{3}0$ ⟨2⟩<br>$2d$ 3.2 $\frac{2}{3}\frac{1}{3}\frac{1}{2}$ | 5 | 2 | $P6_322(2c)$ | **HBP4-3²** |
| $P6_222$ | ..2 | 6-gon | $3a$ 222 000<br>$3a$ 222 11 $-\frac{1}{3}$ | $3c$ 222 $\frac{1}{2}00$<br>$3c$ 222 $1\frac{1}{2}$ $-\frac{1}{3}$ | $3a$ 222 110 ⟨2⟩<br>$3a$ 222 00 $-\frac{1}{3}$ ⟨2⟩ | 7 | 2 | $P6_422(2c)$ | **Q-3²** |
| $P4/mcc$ | $m$.. | 6-gon | $2a$ 422 $00\frac{1}{4}$<br>$2a$ 422 $00\frac{3}{4}$ | $4f$ 222. $\frac{1}{2}0\frac{1}{4}$<br>$2c$ 422 $\frac{1}{2}\frac{1}{2}\frac{3}{4}$ ⟨2⟩ | $4f$ 222. $\frac{1}{2}0\frac{3}{4}$<br>$2c$ 422 $\frac{1}{2}\frac{1}{2}\frac{1}{4}$ | 7 | 2 | $P4/mnc(v)$ | **TAD1-3²** |
| $P422$ | 1 | 5-gon | $1a$ 422 000<br>$1d$ 422 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ ⟨2⟩ | $2e$ 222. $\frac{1}{2}00$<br>$1c$ 422 $\frac{1}{2}\frac{1}{2}0$ | $2f$ 222. $\frac{1}{2}0\frac{1}{2}$ | 7 | 2 | $I422(v, 2c)$ | **TAD2-3²** |
| $P4_222$ | 1 | 8-gon | $2a$ 222. 000 ⟨2⟩<br>$2f$ 2.22 $\frac{1}{2}\frac{1}{2}\frac{1}{4}$<br>$2d$ 222. $0\frac{1}{2}\frac{1}{2}$ | $2d$ 222. $\frac{1}{2}00$<br>$2e$ 2.22 $00\frac{1}{4}$<br>$2c$ 222. $0\frac{1}{2}0$ | $2b$ 222. $\frac{1}{2}\frac{1}{2}0$<br>$2a$ 222. $00\frac{1}{2}$ ⟨2⟩ | 3 | 2 | $P4_322(2c)$ | **TS1-3²** |
| $P422$ | 1 | 6-gon | $1a$ 422 000 ⟨2⟩<br>$1d$ 422 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ ⟨2⟩ | $1b$ 422 $00\frac{1}{2}$<br>$1c$ 422 $\frac{1}{2}\frac{1}{2}0$ | $2f$ 222. $0\frac{1}{2}\frac{1}{2}$<br>$1a$ 422 010 ⟨2⟩ | 7 | 4 | $I4_122(v, 4c)$ | **TS2-3⁴** |
| $P422$ | 1 | 6-gon | $1a$ 422 000 ⟨2⟩<br>$1d$ 422 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ ⟨2⟩ | $1b$ 422 $00\frac{1}{2}$<br>$1c$ 422 $\frac{1}{2}\frac{1}{2}0$ | $2f$ 222. $\frac{1}{2}0\frac{1}{2}$<br>$1a$ 422 010 ⟨2⟩ | 7 | 4 | $I4_122(v, 4c)$ | **TS3-3⁴** |
| $I422$ | ..2 | 6-gon | $2a$ 422 000 ⟨2⟩<br>$2a$ 422 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ ⟨2⟩ | $2a$ 422 100<br>$2a$ 422 $\frac{1}{2}$ $-\frac{1}{2}\frac{1}{2}$ | $2b$ 422 $\frac{1}{2}\frac{1}{2}0$ ⟨2⟩<br>$2b$ 422 $00\frac{1}{2}$ ⟨2⟩ | 7 | 8 | $I4_122(2a, 2c)$ | **TS4-3⁸** |

Table 12. *Non-orientable spanning minimal surfaces with self-intersections entirely along straight lines not subdividing $R^3$ into labyrinths*

| surface sym. $G$ | disc-like surface patch: site sym. | polygon | vertices | | | $l$ | spatial subunits: period. | sym. $U$ | minimal surface |
|---|---|---|---|---|---|---|---|---|---|
| $I422$ | 1 | 7-gon | $2a$ 422 000 ⟨2⟩ | $4c$ 222. $\frac{1}{2}00$ ⟨2⟩ | $4d$ 2.22 $\frac{1}{2}0\frac{1}{4}$ | 5 | 2 | $P2_12_1(2)$. | **TL1-2** |
| | | | $4d$ 2.22 $0\frac{1}{2}\frac{1}{4}$ ⟨2⟩ | $4c$ 222. $0\frac{1}{2}\frac{1}{2}$ ⟨2⟩ | $2a$ 422 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ | | | | |
| | | | $2b$ 422 $00\frac{1}{2}$ ⟨2⟩ | | | | | | |
| $P6_222$ | 1 | 5-gon | $3a$ 222 000 | $3c$ 222 $\frac{1}{2}00$ | $3a$ 222 110 ⟨2⟩ | 11 | 1 | $P(2)2_1(2)$ | **HT1-1** |
| | | | $3b$ 222 11 $-\frac{1}{6}$ ⟨2⟩ | $3b$ 222 00 $-\frac{1}{6}$ ⟨2⟩ | | | | | |
| $Pm\bar{3}n$ | $m$.. | 5-gon | $6d$ $\bar{4}m.2$ $0\frac{1}{4}\frac{1}{2}$ ⟨2⟩ | $\underline{8e\ .32\ \frac{1}{4}\frac{1}{4}\frac{1}{4}}$ ⟨2⟩ | $6d$ $\bar{4}m.2$ $\frac{1}{4}\frac{1}{2}0$ ⟨2⟩ | 7 | 0 | $\bar{4}m.2$ | **WI-00** |
| | | | $\underline{8e\ .32\ \frac{1}{4}\frac{3}{4}\frac{1}{4}}$ ⟨2⟩ | $6d$ $\bar{4}m.2$ $0\frac{3}{4}\frac{1}{2}$ ⟨2⟩ | | | 0 | $m\bar{3}$. | |

along the corresponding edge. Branch points are underlined. The periodicity and the symmetry group $U$ of the spatial subunits are given in the next columns. In the last column the family of self-intersecting minimal surfaces is symbolized. Note that in case of two labyrinths the symmetry group $U$ of a labyrinth, i.e. the symmetry of the surface with coloured labyrinths, coincides with the symmetry group $S$ of the coloured surface for both examples in table 10. A surface patch of an HAT2-$3^2$ surface has been described before by Karcher (1989, p. 308), whereas polygon number II of Stessmann (1934) generates a CAB-00 surface.

Fifteen families of non-orientable surfaces subdividing $R^3$ into $i$ labyrinths are described in table 11. In addition to the information given in table 10, the length $l$ of the shortest Möbius strip is shown (cf. § 3 *a* (iii)). Minimal surfaces belonging to the last three families subdivide $R^3$ into four or eight labyrinths, respectively. The groups of colour permutations corresponding to the respective group–subgroup pairs $G$–$U$ have order 8 (isomorphic to 422) and 64 (not isomorphic to a crystallographic group), respectively. Surface patches described before are those of the WF-$3^2$ surface (Stessmann 1934, polygon VI), of the CBP1-$3^2$ surface (Schoen 1970), and of the TAD2-$3^2$ surface (Karcher 1989, p. 308).

Table 12 gives the analogous information on three families of non-orientable surfaces without 3-periodic labyrinths. As in table 10, the spatial subunits are described by their periodicity and their symmetry group $U$. The surface patch of the HT1-1 surface has been proposed by Karcher (1989, p. 330).

Two of the 20 new families of self-intersecting minimal surfaces, the WI-10 and the WI-00 surfaces, have been described explicitly by Fischer & Koch (1996). Details of the remaining surfaces will be published separately.

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